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A B S T R A C T

We consider a multi-echelon joint inventory-location (MJIL) problem that makes location, order assignment, and inventory decisions simultaneously. The model deals with the distribution of a single commodity from a single manufacturer to a set of retailers through a set of sites where distribution centers can be located. The retailers face deterministic demand and hold working inventory. The distribution centers order a single commodity from the manufacturer at regular intervals and distribute the product to the retailers. The distribution centers also hold working inventory representing product that has been ordered from the manufacturer but has not been yet requested by any of the retailers. Lateral supply among the distribution centers is not allowed. The problem is formulated as a nonlinear mixed-integer program, which is shown to be NP-hard. This problem has recently attracted attention, and a number of different solution approaches have been proposed to solve it. In this paper, we present a Lagrangian relaxation-based heuristic that is capable of efficiently solving large-size instances of the problem. A computational study demonstrates that our heuristic solution procedure is efficient and yields optimal or near-optimal solutions.

## 1. Introduction

Supply chain management (SCM) involves a group of organizations that perform the various processes that are required to manage the flow of products at the lowest possible cost and highest degree of customer satisfaction. The chain typically begins with raw materials and ends with the finished product that is delivered to the customer. The supply chain includes the manufacturer, transporters, warehouses, retailers, and customers themselves. Within each organization, the supply chain includes all functions involved in satisfying customer demand. Supply chain decision phases are classified into the following three categories based on the frequency with which they are made and the time frame over which a decision phase has an impact: (1) strategic decisions, which impact the firm over several years, for example the locations of distribution centers; (2) tactical decisions, which are usually made one to four times a year, for example determining transportation and inventory policies; and (3) operational decisions, which are usually made on a daily basis, for example scheduling and routing decisions; see Chopra and Meindl [6] and Simchi-Levi et al. [31]. In today's competitive environment, only efficient supply chains that integrate decisions in the various phases can survive.

Inventory management and facility location are two major issues in the efficient design of a supply chain network; see Gunasekaran et al. [16,17] and Stevens [34]. However, literature on supply chain optimization has traditionally considered these issues independently not only because of different planning horizons but principally because of the computational complexity of the joint optimization problem. Indeed, facility location problems are typically NP-hard combinatorial optimization problems, and the majority of inventory management problems are formulated as nonlinear programming problems. Combining such two problems leads to more difficult NP-hard problems that are usually nonconvex, and therefore cannot be easily solved to optimality using exact optimization methods. However, such an integration offers a possibility to considerably improve the supply chain management and reduce the costs.

It is worth mentioning a real-world example, in the interest of demonstrating how the strategic level decision of facility location (which does not necessarily refer to an actual location of a new facility) can be successfully integrated with the tactical level inventory decisions, to provide better solutions and lead to improved performance. The motivation behind initial work on joint inventory location problems arose from the problem of

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producing and distributing blood platelets for a blood bank in Chicago and it was addressed by Daskin et al. [8], Shen et al. [23] and Ozsen et al. [27,28]. This particular bank distributes blood to more than thirty hospitals in the region and the inventory cost of the platelets is high, due to specific conditions that should be maintained at all times, such as the frequent agitation of the platelets or the temperature that must be kept between 20 and 24 °C. Furthermore, the expiration of the blood platelets a few days after they are collected is another important factor to be considered. Each hospital stored its own platelet inventory and this independent inventory and location policy led to platelets going to waste after expiration in certain hospitals, while others ran out very soon. After re-addressing this problem as a joint location-inventory model, by enforcing some hospitals to serve as distribution centers and others as retailers, the efficiency and usage of the platelets greatly improved. The same example motivated Le et al. [22] to jointly study inventory decisions and routing decisions for perishable goods. Therefore, many real-world problems can be formulated as location-inventory problems without including any "real" location decisions for opening new warehouses. The existence of the location-type decision variables, in addition to the inventory decision variables, is the only reason for calling such problems location-inventory problems.

The objective of the present paper is to develop an efficient Lagrangian-based heuristic for such an integrated problem, which we call a large-scale multi-echelon joint inventory-location problem and which we refer to as the MJIL problem.

The paper is organized as follows: Section 2 reviews existing joint inventory-location models; Section 3 introduces the MJIL problem; Section 4 introduces the new Lagrangian relaxation algorithm for solving the MJIL problem; Section 5 presents computational studies; and Section 6 discusses future research directions.

## 2. Literature review

Integrated supply chain network design involves several core components, among which are facility location and inventory management. Most literature on supply chain optimization has traditionally considered facility location decisions and inventory management decisions independently: Amiri [1], Daskin et al. [9], Hindi and Pienkosz [18], Pirkul and Jayaraman [29], and Tsiakis et al. [40] focused on location decisions, while Axsäter [2], Jones and Riley [20], Muckstadt and Roundy [25], Svoronos and Zipkin [35], and Wee and Yang [41] focused on inventory management decisions. Only recently, integrated models have attracted the attention of researchers.

Barahona and Jensen [3] introduced a large-scale integer programming formulation for a location-inventory model, and used Dantzig–Wolfe decomposition to solve the linear programming relaxation of this problem. Because the standard implementation of the Dantzig-Wolfe decomposition algorithm was too slow, the authors used subgradient optimization to improve the rate of convergence of their solution procedure. Although they included ordering and inventory costs in their model, they considered these costs only for one echelon. Thus, their model represents the integration of a location model with an economic order quantity (EOQ) model; see Nahmias [26].

Erlebacher and Meller [11] developed an analytical joint location-inventory model. The general version of their problem is NP-hard, and is therefore difficult to solve, so they developed a heuristic algorithm which performs well on their test problems. They consider ordering and inventory costs at the distribution centers but these costs are omitted at the retailer level. Teo et al. [37] used an analytical modeling approach to study the impact on facility investments and inventory costs when several distribution

**Table 1**
Comparison of relevant published papers.

| Paper | Model features | Decision variables | Solution methodology |
|---|---|---|---|
| This paper | • Ordering, inventory and transportation costs<br>• Multi-echelon<br>• Single sourcing | • Average inventory level at retailer<br>• Average inventory level at DC<br>• Order-quantity at retailer<br>• Cycle time of retailer<br>• Cycle-time of DC | Lagrangian relaxation-based heuristic |
| [3] | • Ordering and inventory costs<br>• Single echelon | • Whether or not a plant is opened<br>• Customer-plant assignment<br>• Whether or not a customer that is assigned to a plant requires a certain part | Dantzig–Wolfe decomposition and subgradient optimization |
| [11] | • Ordering and inventory costs<br>• Two-level distribution system | • Number of DCs<br>• Location of DCs<br>• If DC is open<br>• If DC serves certain customer grid<br>• Average distance from DC to customer<br>• Demand shipped<br>• Distance from plant to DC | Stylized analytical model, heuristics |
| [37] | • Stochastic demands at customer locations<br>• Warehouse consolidation | • Location of DC<br>• Assignment of demand location to DC | Consolidation strategy |
| [36] | • Possibility for direct flow between customer and factory<br>• Multiple sourcing | • Flows between customers, factories, DCs<br>• Size of shipments sent<br>• Total flow passing from every DC | Iterative heuristic |
| [4] | • Two stage distribution system<br>• Explicit modeling of inventory replenishment, holding and transportation costs | • Set of open DCs<br>• Assignment of open DCs | Analytical solution |
| [39] | • Multiple sourcing | • Location of DCs<br>• Assignment of retailers to DCs | Continuous approximation |

centers are consolidated into a central distribution center, but did not consider transportation costs in their formulation.

Transportation costs are considered in most recent models, as found in the work of Tancrez et al. [36], in which the authors study the integrated location-inventory problem for three-level supply networks, comprising suppliers, distribution centers and retailers. The non-linear continuous formulation includes transportation, fixed, handling and inventory holding costs and the authors develop a heuristic to solve the problem, which performs efficiently. Keskin and Üster [21] similarly address the three-stage supply network, in terms of modeling assumptions and considerations; however, as a solution methodology they develop both a local search and a Simulated Annealing (SA) algorithm and conclude that SA leads to better quality solutions and lower run times. As in the current paper, both aforementioned works assume single sourcing.

Çetinkaya et al. [4] further consider that transportation costs are subject to truck and cargo capacity, leading to a need for explicit cargo cost modelling. On the other hand, they consider a two-stage distribution system with DCs and retailers and they take advantage of certain structural properties to reduce to a simpler non-linear formulation, which leads to efficient solving of the problem. Contrary to the discrete models seen in the papers mentioned thus far, Tsao et al. [22] develop a continuous approximation approach, with the motivation of solving larger-scale problems. They conduct a sensitivity analysis with respect to parameter values to provide management insights.

Daskin et al. [8] and Shen et al. [23] developed a location-inventory model with risk pooling (LRMP). LMRP is formulated as a nonlinear integer programming problem that incorporates inventory costs at the distribution centers. Ozsen et al. [27,28] and Sourirajan et al. [32,33] proposed two different extensions to the model, presented by Daskin et al. [8] and Shen et al. [23]. Table 1 provides a straightforward comparison of several of the published papers, in terms of distinct model assumptions, decisions made and solution methodology.

In this paper we study a model that considers ordering and inventory costs at both the distribution center and retailer echelons. This model, which was studied by Teo and Shu [38], Shu [30], and Diabat et al. [10], is unlike the models mentioned above, which consider the ordering and inventory costs at only one level of the supply chain network. Considering ordering and inventory costs at two echelons in the model poses additional challenges. Teo and Shu [38] developed a column generation-based algorithm for solving this model that was capable of finding an optimal or near-optimal solution for small to moderate size instances. Shu [30] presented a greedy heuristic to solve large-scale instances of the problem. Later on, Diabat et al. [10] developed a basic Lagrangian relaxation-based heuristic for this problem. Results from the aforementioned works that employ Lagrangian relaxation-based heuristics demonstrate that this method seems promising for such integrated problems. Therefore, in this paper, a new sophisticated Lagrangian relaxation-based heuristic with efficient lower and upper bounding schemes is presented. Our computational studies demonstrate that the proposed method is efficient, and yields optimal or near-optimal solutions in relatively very small computational times.

## 3. Problem formulation

The formulation addresses the delivery of a single product from a manufacturer to distribution centers, which can be opened in multiple locations, and from there to multiple retailers. Single sourcing is assumed, according to which a single distribution center covers the total demand of any given retailer. On the retailers' side the demand is deterministic and they hold working inventory, which is defined as the product delivered to the retailer by the distribution center, without having yet been requested by end-customers. As far as distribution centers are concerned, they receive a single commodity only from the manufacturer, as lateral supply is prohibited, at frequent time periods, in order to then redistribute to retailers, and they also hold working inventory of product from the manufacturer, that has not yet been ordered from retailers. Fig. 1 serves as a representation of the described system.

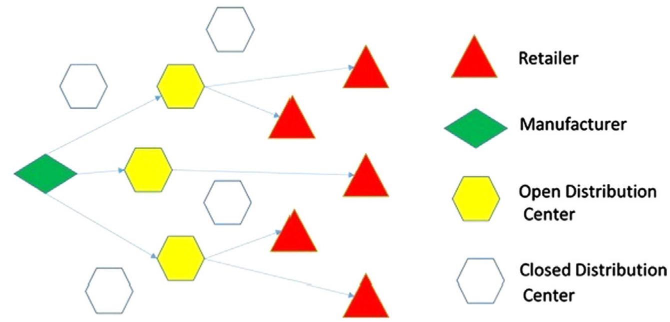


**Fig. 1.** MJIL supply chain network.

There are four main cost components in this system: (i) fixed-order cost: the cost of placing an order, independent of the size of the order, (ii) unit-inventory cost: the cost of holding one unit of commodity for one unit of time, (iii) unit-shipping cost: the cost of shipping one unit of commodity between facilities, and (iv) fixed-location cost: the cost associated with establishing and operating a distribution center. The objective of the formulation is to decide: (1) the number of distribution centers to establish; (2) their locations; (3) the sets of retailers assigned to each distribution center; and (4) the size and timing of orders for each facility, with the aim of minimizing the sum of inventory, shipping, ordering, and location costs while satisfying end-customer demand.

To formulate the problem, Diabat et al. [10] introduced the following notation:

*Sets*

| | |
|---|---|
| $I$ | set of retailers, indexed by $i$ |
| $J$ | set of potential distribution center locations, indexed by $j$ |
| $\mathfrak{I}_j$ | subset of retailers that are assigned to the distribution center at location $j$ |

*Parameters*

| | |
|---|---|
| $d_i$ | demand rate of retailer $i$ |
| $f_j$ | fixed cost of establishing and operating a distribution center at location $j$ |
| $s_{ij}$ | unit-shipping cost to retailer $i$ from distribution center $j$ |
| $\hat{s}_j$ | unit-shipping cost from the manufacturer to distribution center $j$ |
| $h_i$ | unit-inventory cost per unit of time at retailer $i$ |
| $k_i$ | fixed-order cost at retailer $i$ |
| $\hat{h}_j$ | unit-inventory cost at distribution center $j$, per unit of time |
| $\hat{k}_j$ | fixed-order cost at distribution center $j$ |
| $t_B$ | base-planning period |
| $\beta_{trn}$ | weight factor associated with transportation costs, $\beta_{trn} \geq 0$ |
| $\beta_{inv}$ | weight factor associated with inventory costs, $\beta_{inv} \geq 0$ |

Decision variables

| | |
|---|---|
| $V_i$ | average inventory level at retailer $i$ |
| $\widehat{V}_j$ | average inventory level at distribution center $j$ |

| | | |
|---|---|---|
| $Q_i$ | order-quantity at retailer $i$ | |
| $T_{ij}$ | cycle-time of retailer $i$ when served by distribution center $j$ | |
| $\widehat{T}_j$ | cycle-time of distribution center $j$ | |

*Binary decision variables*

$X_j$ $\begin{cases} 1 & \text{if a distribution center is opened at candidate location } j \\ 0 & \text{otherwise} \end{cases}$

$Y_{ij}$ $\begin{cases} 1 & \text{if retailer } i \text{ is served by the distribution center at location } j \\ 0 & \text{otherwise.} \end{cases}$

Now, we can formulate the MJIL problem as follows:

$$\begin{aligned}
\min_{T,\widehat{T},X,Y} \quad & \sum_{j \in J} f_j X_j + \sum_{j \in J}\sum_{i \in I} \beta_{trn}(s_{ij}+\hat{s}_j)d_i Y_{ij} + \sum_{j \in J}\sum_{i \in I} \beta_{inv}\frac{k_i}{T_{ij}}Y_{ij} \\
& + \sum_{j \in J} \beta_{inv}\frac{\hat{k}_j}{\widehat{T}_j}X_j + \sum_{j \in J}\sum_{i \in I}\frac{1}{2}\beta_{inv}(h_i-\hat{h}_j)d_i T_{ij}Y_{ij} \\
& + \sum_{j \in J}\sum_{i \in I}\beta_{inv}\frac{1}{2}\hat{h}_j d_i \max\{T_{ij},\widehat{T}_j\}Y_{ij} \\
& = \sum_{j \in J}\left(f_j + \beta_{inv}\frac{\hat{k}_j}{\widehat{T}_j}\right)X_j \\
& + \sum_{j \in J}\sum_{i \in I}\left(\beta_{trn}b_{ij} + \beta_{inv}c_{ij}T_{ij} + \beta_{inv}\frac{k_i}{T_{ij}} + \beta_{inv}e_{ij} \max\{T_{ij},\widehat{T}_j\}\right)Y_{ij}
\end{aligned} \tag{1}$$

$$\text{s.t.} \quad \sum_{j \in J} Y_{ij} = 1, \quad \forall i \in I \tag{2}$$

$$Y_{ij} - X_j \le 0, \quad \forall i \in I, j \in J \tag{3}$$

$$T_{ij} \in \mathbb{R}_+, \quad \forall i \in I, j \in J \tag{4}$$

$$\widehat{T}_j \in \mathbb{R}_+, \quad \forall j \in J \tag{5}$$

$$X_j \in \{0,1\}, \quad \forall j \in J \tag{6}$$

$$Y_{ij} \in \{0,1\}, \quad \forall i \in I, j \in J \tag{7}$$

where (1) represents the objective function that minimizes the sum of inventory, shipping, ordering, and location costs while satisfying end-customer demand. We have defined $b_{ij} = (s_{ij}+\hat{s}_j)d_i$, $c_{ij} = \frac{1}{2}(h_i - \hat{h}_j)d_i$, and $e_{ij} = \frac{1}{2}\hat{h}_j d_i$. Observe that even when retailer $i$ is not assigned to a distribution center at location $j$, the variable $T_{ij}$ will be assigned a value. However, this value does not play a role in the objective function as it is multiplied by $Y_{ij} = 0$.

According to constraint (2), each retailer has to be assigned to exactly one distribution center. Constraint (3) assures that a retailer can be assigned to a distribution center only if it is opened. Constraints (4) and (5) define variables $T_{ij}$ and $\widehat{T}_j$ as positive real numbers. Constraints (6) and (7) define variables $X_j$ and $Y_{ij}$ as binary numbers. For more explanation on the formulation of the problem, we refer the reader to Diabat et al. [10].

Let $\mathfrak{I}_j$ be the set of retailers that are assigned to distribution center $j$. Based on the results obtained in [10], we define sub-problem($j$) to be the system consisting of the distribution center $j$ and the set of retailers $\mathfrak{I}_j$. Because of our single-sourcing assumption, the problem decomposes into $|J|$ sub-problems, each representing a one-distribution center multi-retailer inventory system. The goal of each sub-problem is to find an optimal inventory policy, that is, the size and timing of orders for each facility, so as to minimize the sum of ordering and inventory costs while meeting demand. To find the subsets $\mathfrak{I}_j, \forall j \in J$, each sub-problem is defined as nonlinear program as shown in [10] and should be solved endogenously and simultaneously with problem (1)–(7), since its decisions are interrelated with the ordering decisions $T_{ij}$ and $\widehat{T}_j$.

If $\beta_{inv} = 0$, the MJIL problem (1)–(7) reduces to the uncapacitated fixed-charge location problem (UFLP); see Daskin [7]. Therefore, the MJIL problem is NP-hard. In fact, the nonconvexity of (1)–(7) indicates that it is probably difficult to solve the problem to global optimality. We now propose an alternative formulation for the nonlinear mixed integer program (1)–(7) that is easier to work with. For simplicity, we drop the index $j$ on the retailer cycle-time, that is, $T_{ij}$ is replaced by $T_i$. We define $Z_j(\mathfrak{I}_j, \widehat{T}_j, T_i, X_j)$ and $Z_j^*(\mathfrak{I}_j, X_j)$ to respectively be the total average inventory cost and the optimal inventory cost of serving all retailers in $\mathfrak{I}_j$ from a distribution center located at location $j$. Formally, we have

$$Z_j(\mathfrak{I}_j, \widehat{T}_j, T_i, X_j) = \frac{\hat{k}_j}{\widehat{T}_j}X_j + \sum_{i \in \mathfrak{I}_j}\left(\frac{k_i}{T_i} + c_{ij}T_i + e_{ij}\max(T_i, \widehat{T}_j)\right) \tag{8}$$

and

$$Z_j^*(\mathfrak{I}_j, X_j) = \min_{\widehat{T}_j, T_i}\left\{\begin{array}{l} \frac{\hat{k}_j}{\widehat{T}_j}X_j + \sum_{i \in \mathfrak{I}_j}\left(\frac{k_i}{T_i} + c_{ij}T_i + e_{ij}\max(T_i,\widehat{T}_j)\right) \\ \text{s.t.} \quad \widehat{T}_j \in \mathbb{R}_+ \\ \qquad T_i \in \mathbb{R}_+, \quad \forall i \in \mathfrak{I}_j \end{array}\right\}. \tag{9}$$

By convention we interpret $Z_j^*(\mathfrak{I}_j)$ as $Z_j^*(\mathfrak{I}_j, X_j = 0)$ when $\mathfrak{I}_j = \emptyset$ and as $Z_j^*(\mathfrak{I}_j, X_j = 1)$ when $\mathfrak{I}_j \neq \emptyset$.

With this convention, the program (1)–(7) can be reformulated as follows:

$$\min_{X,Y,\mathfrak{I}} \quad \sum_{j \in J}\left(f_j X_j + \sum_{i \in I}\beta_{trn}b_{ij}Y_{ij} + \beta_{inv}Z_j^*(\mathfrak{I}_j)\right) \tag{10}$$

$$\text{s.t.} \quad \sum_{j \in J} Y_{ij} = 1, \quad \forall i \in I \tag{11}$$

$$Y_{ij} - X_j \le 0, \quad \forall i \in I, j \in J \tag{12}$$

$$X_j \in \{0,1\}, \quad \forall j \in J \tag{13}$$

$$Y_{ij} \in \{0,1\}, \quad \forall i \in I, j \in J \tag{14}$$

where $\mathfrak{I}_j = \{i \in I | Y_{ij} = 1\}$.

## 4. Solution approach

Lagrangian relaxation has shown exceptional success in solving many NP-hard supply chain combinatorial optimization problems; see for example Chen and Chu [5], Eskigun et al. [12], Jayaraman and Pirkul [19], Min et al. [24], and Pirkul and Jayaraman [29]. Excellent surveys of the computational aspects and applications of Lagrangian relaxations are given by Fisher [13–15]. In the next two subsections, we describe our approach to solving problem (10)–(14). The proposed solution procedures are based on Lagrangian dual formulations, where Constraints (11) are relaxed. A solution of the Lagrangian dual provides a lower bound on the program (10)–(14). In order to find an upper bound, we use a heuristic that constructs a feasible solution from the lower bound solution.

### 4.1. Lower bounds

The Lagrangian dual obtained by relaxing constraints (11) can be written as

$$\begin{aligned}
\max_{\lambda \ge 0}\min_{X,Y,\mathfrak{I}} \quad & \sum_{j \in J}\left(f_j X_j + \sum_{i \in I}\beta_{trn}b_{ij}Y_{ij} + \beta_{inv}Z_j^*(\mathfrak{I}_j)\right) + \sum_{i \in I}\lambda_i\left(1 - \sum_{j \in J}Y_{ij}\right) \\
& = \sum_{j \in J}\left(f_j X_j + \sum_{i \in I}(\beta_{trn}b_{ij} - \lambda_i)Y_{ij} + \beta_{inv}Z_j^*(\mathfrak{I}_j)\right) + \sum_{i \in I}\lambda_i
\end{aligned} \tag{15}$$

$$\text{s.t.}\quad Y_{ij}-X_j\le 0,\quad \forall i\in I, j\in J \tag{16}$$

$$X_j\in\{0,1\},\quad \forall j\in J \tag{17}$$

$$Y_{ij}\in\{0,1\},\quad \forall i\in I, j\in J. \tag{18}$$

Problem (15)–(18) decomposes by location, that is, each distribution center location can be considered separately. Denoting the corresponding sub-problem $S_j(\lambda)$ and defining its corresponding optimal value as $S_j^*(\lambda)$, we obtain

$$\min_{X,Y,\mathfrak{I}}\ f_jX_j+\sum_{i\in I}(\beta_{trn}b_{ij}-\lambda_i)Y_{ij}+\beta_{inv}Z_j^*(\mathfrak{I}_j) \tag{19}$$

$$\text{s.t.}\quad Y_{ij}-X_j\le 0,\quad \forall i\in I \tag{20}$$

$$X_j\in\{0,1\} \tag{21}$$

$$Y_{ij}\in\{0,1\},\quad \forall i\in I. \tag{22}$$

For any given $\lambda$, we either set $X_j=0$ or $X_j=1$ in problem $S_j(\lambda)$. If we set $X_j=0$, then $Y_{ij}=0$ for all $i\in I$, so that $\mathfrak{I}_j=\emptyset$ (empty set), and hence $Z_j^*(\mathfrak{I}_j)=0$. Therefore, $S_j^*(\lambda)=0$. On the other hand, if we set $X_j=1$, then for each $i\in I$ we can select either $Y_{ij}=0$ ($i\notin\mathfrak{I}_j$) or $Y_{ij}=1$ ($i\in\mathfrak{I}_j$). Since $Z_j^*(\mathfrak{I}_j)\ge Z_j^*(\mathfrak{I}_j\backslash\{i\})$ for any $\mathfrak{I}_j\subseteq J$, when $\beta_{trn}b_{ij}-\lambda_i>0$ it is never advantageous to set $Y_{ij}=1$ for any $i$, that is, we must select $Y_{ij}=0$, so that $i\notin\mathfrak{I}_j$.

A more challenging decision to make occurs when $X_j=1$ and $\beta_{trn}b_{ij}-\lambda_i<0$. The structure of subproblem $S_j(\lambda)$, (19)–(22), makes it difficult to decide if it is beneficial to set $Y_{ij}=1$ or not, for any $i\in I$. If the term $Z_j^*(\mathfrak{I}_j)$ of the objective function was not present in (19), as in the UFLP, then we would simply set $Y_{ij}=1$ whenever $X_j=1$ and $\beta_{trn}b_{ij}-\lambda_i\le 0$. However, the existence of $Z_j^*(\mathfrak{I}_j)$ will force the objective value $S_j^*(\lambda)$ to increase since setting $Y_{ij}=1$ implies $i\in\mathfrak{I}_j$. Therefore, although $X_j=1$ and $\beta_{trn}b_{ij}-\lambda_i\le 0$, we might have to set $Y_{ij}=0$ if the increase in the optimal objective value is greater than $(\beta_{trn}b_{ij}-\lambda_i)$ for any $i$ in $\mathfrak{I}_j$. It is therefore important to determine how much of an increase in $S_j^*(\lambda)$ will be caused by adding $i$ to $\mathfrak{I}_j$.

**Definition 4.1.** The $j$th marginal inventory cost of retailer $i$, denoted by $M_i(\mathfrak{I}_j)$ for any $i\in\mathfrak{I}_j$, is the difference in the optimal average inventory costs of sub-problem($j$) between serving retailer $i$ or not. That is, $M_i(\mathfrak{I}_j)=Z_j^*(\mathfrak{I}_j)-Z_j^*(\mathfrak{I}_j\backslash\{i\})$ for any $i\in\mathfrak{I}_j$ and any $\mathfrak{I}_j\subseteq I$ where $Z_j^*(\emptyset)=0$.

**Proposition 4.1.** *Let $\mathfrak{I}_j\subseteq I$ and let $i\in\mathfrak{I}_j$. The jth marginal inventory cost of retailer i is lower bounded by $\sqrt{2k_id_ih_i}$; i.e. $M_i(\mathfrak{I}_j)\ge\sqrt{2k_id_ih_i}$.*

**Proof of Proposition 4.1.** Assume $\widehat{T}_j$ is fixed. Define

$$g_{ij}(T_i,\widehat{T}_j)=\frac{k_i}{T_i}+\frac{1}{2}(h_i-\hat{h}_j)d_iT_i+\frac{1}{2}\hat{h}_jd_i\max(T_i,\widehat{T}_j). \tag{23}$$

Let $\partial g_{ij}(T_i,\widehat{T}_j)$ be the subgradient of $g_{ij}$ with respect to its first argument, which we will abbreviate as $\partial g_{ij}(T_i)$. Clearly, the function $g_{ij}(T_i,\widehat{T}_j)$ is a convex function of its first argument over $T_i>0$ since it is the sum of three functions that are convex in $T_i$ for fixed $\widehat{T}_j$; it follows that the minimum of $g_{ij}$ over $T_i>0$ will be attained at any point where $0\in\partial g_{ij}$. We have

$$\partial g_{ij}(T_i,\widehat{T}_j)=\begin{cases}-\dfrac{k_i}{T_i^2}+\dfrac{1}{2}(h_i-\hat{h}_j)d_i & \text{if } T_i<\widehat{T}_j\\ -\dfrac{k_i}{T_i^2}+\dfrac{1}{2}(h_i-\hat{h}_j)d_i,\ -\dfrac{k_i}{T_i^2}+\dfrac{1}{2}h_id_i & \text{if } T_i=\widehat{T}_j\\ -\dfrac{k_i}{T_i^2}+\dfrac{1}{2}h_id_i & \text{if } T_i>\widehat{T}_j\end{cases} \tag{24}$$

There are three possibilities:

Case 1: If $\widehat{T}_j\le\sqrt{2k_i/h_id_i}$, then define $T_i^*=\sqrt{2k_i/d_ih_i}$. It then follows that $T_i^*>\widehat{T}_j$ and hence that $\partial g_{ij}(T_i^*)=0$, and thus that $T_i^*$ minimizes $g_{ij}(\cdot,\widehat{T}_j)$.

Case 2: If $\widehat{T}_j\ge\sqrt{2k_i/(h_i-\hat{h}_j)d_i}$, then define $T_i^*=\sqrt{2k_i/(h_i-\hat{h}_j)d_i}$. It then follows that $T_i^*<\widehat{T}_j$ and hence that $\partial g_{ij}(T_i^*)=0$, and thus that $T_i^*$ minimizes $g_{ij}(\cdot,\widehat{T}_j)$.

Case 3: If $\sqrt{2k_i/h_id_i}\le\widehat{T}_j\le\sqrt{2k_i/(h_i-\hat{h}_j)d_i}$, then define $T_i^*=\widehat{T}_j$. It then follows that $0\in\partial g_{ij}(T_i^*)$, and thus that $T_i^*$ minimizes $g_{ij}(\cdot,\widehat{T}_j)$.

It follows from the above that $g_{ij}(\cdot,\widehat{T}_j)$ attains a minimum at

$$T_i^*=\begin{cases}\sqrt{\dfrac{2k_i}{d_ih_i}} & \text{if } \widehat{T}_j\le\sqrt{\dfrac{2k_i}{h_id_i}}\\ \widehat{T}_j & \text{if } \sqrt{\dfrac{2k_i}{h_id_i}}\le\widehat{T}_j\le\sqrt{\dfrac{2k_i}{(h_i-\hat{h}_j)d_i}}\\ \sqrt{\dfrac{2k_i}{(h_i-\hat{h}_j)d_i}} & \text{if } \widehat{T}_j\ge\sqrt{\dfrac{2k_i}{(h_i-\hat{h}_j)d_i}}\end{cases} \tag{25}$$

Let

$$g_i(\widehat{T}_j)=\min_{T_i>0}g_{ij}(T_i,\widehat{T}_j)$$

be the value of $g_{ij}$ at the minimum, then

$$g_i(\widehat{T}_j)=\begin{cases}\sqrt{2k_id_ih_i} & \text{if } \widehat{T}_j<\sqrt{\dfrac{2k_i}{d_ih_i}}\\ \dfrac{k_i}{\widehat{T}_j}+\dfrac{1}{2}d_ih_i\widehat{T}_j & \text{if } \sqrt{\dfrac{2k_i}{d_ih_i}}\le\widehat{T}_j\le\sqrt{\dfrac{2k_i}{(h_i-\hat{h}_j)d_i}}\\ \sqrt{2k_i(h_i-\hat{h}_j)d_i}+\dfrac{1}{2}\hat{h}_jd_i\widehat{T}_j & \text{if } \widehat{T}_j>\sqrt{\dfrac{2k_i}{(h_i-\hat{h}_j)d_i}}.\end{cases} \tag{26}$$

Since $g_i(\widehat{T}_j)$ is continuous and nondecreasing in $\widehat{T}_j$, it follows that

$$\min_{\widehat{T}_j}\{g_i(\widehat{T}_j)\}=\sqrt{2k_id_ih_i}.$$

Without loss of generality, select retailer $i\in\mathfrak{I}_j$. We prove next that $M_i(\mathfrak{I}_j)\ge\sqrt{2k_id_ih_i}$, for any $\mathfrak{I}_j\subseteq I$.

$$\begin{aligned}M_i(\mathfrak{I}_j)&=Z_j^*(\mathfrak{I}_j)-Z_j^*(\mathfrak{I}_j\backslash\{i\})\\ &=\min_{\widehat{T}_j,T_i}\{Z_j(\mathfrak{I}_j,\widehat{T}_j,T_i,X_j)\}-\min_{\widehat{T}_j,T_i}\{Z_j(\mathfrak{I}_j\backslash\{i\},\widehat{T}_j,T_i,X_j)\}\\ &=\min_{\widehat{T}_j,T_i}\left\{\frac{\hat{k}_j}{\widehat{T}_j}X_j+\sum_{i\in\mathfrak{I}_j}\left(\frac{k_i}{T_i}+c_{ij}T_i+e_{ij}\max(T_i,\widehat{T}_j)\right)\right\}\\ &\quad-\min_{\widehat{T}_j,T_i}\left\{\frac{\hat{k}_j}{\widehat{T}_j}X_j+\sum_{i\in\mathfrak{I}_j\backslash\{i\}}\left(\frac{k_i}{T_i}+c_{ij}T_i+e_{ij}\max(T_i,\widehat{T}_j)\right)\right\}\\ &\ge\min_{\widehat{T}_j,T_i}\left\{\frac{k_i}{T_i}+c_{ij}T_i+e_{ij}\max(T_i,\widehat{T}_j)\right\}\\ &\quad+\min_{\widehat{T}_j,T_i}\left\{\frac{\hat{k}_j}{\widehat{T}_j}X_j+\sum_{i\in\mathfrak{I}_j\backslash\{i\}}\left(\frac{k_i}{T_i}+c_{ij}T_i+e_{ij}\max(T_i,\widehat{T}_j)\right)\right\}\\ &\quad-\min_{\widehat{T}_j,T_i}\left\{\frac{\hat{k}_j}{\widehat{T}_j}X_j+\sum_{i\in\mathfrak{I}_j\backslash\{i\}}\left(\frac{k_i}{T_i}+c_{ij}T_i+e_{ij}\max(T_i,\widehat{T}_j)\right)\right\}\end{aligned}$$

$$= \min_{\widehat{T}_j} \left\{ \min_{T_i} \left\{ \frac{k_i}{T_i} + c_{ij} T_i + e_{ij} \max(T_i, \widehat{T}_j) \right\} \right\}$$

$$= \min_{\widehat{T}_j} \left\{ g_i(\widehat{T}_j) \right\}$$

$$= \sqrt{2k_i d_i h_i}.$$

**Proposition 4.2.** *Assume that $\mathfrak{I}_j^* \subseteq I$ is the set of retailers that will be assigned to distribution center $j$ in the optimal solution to problem $S_j(\lambda)$. Then*

$$\beta_{trn} b_{ij} - \lambda_i + \beta_{inv}\sqrt{2k_i d_i h_i} \le 0 \quad \text{for any } i \in \mathfrak{I}_j^*.$$

**Proof of Proposition 4.2.** Since $i^* \in \mathfrak{I}_j^*$, it follows that

$$\sum_{i \in \mathfrak{I}_j^*} (\beta_{trn} b_{ij} - \lambda_i) + \beta_{inv} Z_j^*(\mathfrak{I}_j^*) \le \sum_{i \in \mathfrak{I}_j^* \backslash \{i^*\}} (\beta_{trn} b_{ij} - \lambda_i) + \beta_{inv} Z_j^*(\mathfrak{I}_j^* \backslash \{i\}).$$

This inequality can be simplified to

$$(\beta_{trn} b_{i^*j} - \lambda_{i^*}) + \beta_{inv} Z_j^*(\mathfrak{I}_j^*) \le \beta_{inv} Z_j^*(\mathfrak{I}_j^* \backslash \{i^*\}) \le \beta_{inv} Z_j^*(\mathfrak{I}_j^*) - \beta_{inv}\sqrt{2k_{i^*} d_{i^*} h_{i^*}}.$$

Simplifying the above expression, we obtain

$$0 \le -(\beta_{trn} b_{i^*j} - \lambda_{i^*}) - \beta_{inv}\sqrt{2k_{i^*} d_{i^*} h_{i^*}}$$

or equivalently

$$0 \ge (\beta_{trn} b_{i^*j} - \lambda_{i^*}) + \beta_{inv}\sqrt{2k_{i^*} d_{i^*} h_{i^*}}$$

which is the desired result.

Proposition 4.2 yields the following helpful contrapositive that can be used in designing our solution algorithm for solving the MJIL problem.

**Corollary 1.** *Assume that $\mathfrak{I}_j^* \subseteq I$ is the set of retailers that will be assigned to distribution center $j$ in the optimal solution to problem $S_j(\lambda)$. For any $i \in I$,*

$$\text{if } \beta_{trn} b_{ij} - \lambda_i + \beta_{inv}\sqrt{2k_i d_i h_i} > 0 \quad \text{then } i \notin \mathfrak{I}_j^*. \tag{27}$$

Corollary 1 shows that, in solving subproblem $S_j(\lambda)$, we should choose $Y_{ij} = 0$ for any retailer $i \in I$ that has a positive value of $\beta_{trn} b_{ij} - \lambda_i + \beta_{inv}\sqrt{2k_i d_i h_i}$. Unfortunately, $X_j = 1$ and $\beta_{trn} b_{ij} - \lambda_i + \beta_{inv}\sqrt{2k_i d_i h_i} < 0$ are not sufficient conditions to require $Y_{ij} = 1$ for any $i \in I$. However, Corollary 1 shows that we can restrict our search to the set of retailers:

$$I_j = \{i \in I \ \text{ s.t. } \ \beta_{trn} b_{ij} - \lambda_i + \beta_{inv}\sqrt{2k_i d_i h_i} < 0\}.$$

For those retailers, we sort all $i \in I_j$ such that $\delta_{1j} \le \delta_{2j} \le \cdots \le \delta_{mj}$, where $m = |I_j|$ and $\delta_{ij} = (\beta_{trn} b_{ij} - \lambda_i) + Z_j^*(\{i\})$. Then we use a simple greedy heuristic to assign these retailers as outlined in Algorithm **AlgLB** below. This algorithm is used to obtain a lower bound on $S_j(\lambda)$.

**Algorithm AlgLB.**

*Step* 1: Partition set $I$ into two subsets as follows:

$$I_j^{\notin} = \{i \in I \ \text{ s.t. } \ \beta_{trn} b_{ij} - \lambda_i + \beta_{inv}\sqrt{2kd_i h} \ge 0\}$$
$$I_j = I - I_j^{\notin}$$

*Step* 2: Compute $\delta_{ij} = (\beta_{trn} b_{ij} - \lambda_i) + Z_j^*(\{i\})$, for $i = 1, \ldots, m$.

*Step* 3: Form the sets

$I_j^{(k)} = \{\ell \in I_j : \ \delta_{\ell j}$ is among the $k$ smallest elements of $\{\delta_{ij}\}_{i=1,\ldots,m}\}$.

*Step* 4: Compute the partial sums

$$\Delta_j^{(k)} = \beta_{inv} Z_j^*(I_j^{(k)}) + \sum_{i \in I_j^{(k)}} (\beta_{trn} b_{ij} - \lambda_i) \text{ for } k = 1, 2, \ldots, m.$$

*Step* 5: Let $k^*$ be the value of $k$ that gives the minimum value of $\Delta_j^{(k)}$ and set

$$X_j^* = \begin{cases} 1 & \text{if } f_j + \Delta_j^{(k^*)} < 0 \\ 0 & \text{otherwise.} \end{cases}$$

*Step* 6: Set

$$Y_{ij}^* = \begin{cases} 1 & \text{if } X_j = 1 \text{ and } i \in I_j^{(k^*)} \\ 0 & \text{otherwise.} \end{cases}$$

Algorithm **AlgLB** is applied for every $j \in J$. The objective function (10) evaluated at $X_j^*$, $Y_{ij}^*$ serves as a lower bound on the MJIL problem, (10)–(14).

### *4.2. Upper bounds*

In most cases, the solution obtained by algorithm **AlgLB** will be infeasible for the MJIL problem. However, a feasible solution can typically be obtained by using a constructive heuristic on the lower bound solution, as described in the algorithm **AlgUB**. The resulting feasible solution provides an upper bound on (10)–(14). The problem that algorithm **AlgUB** addresses is: given an initial solution $X$, $Y^{(0)}$ that satisfies (12)–(14) but not necessarily (11), how does one extend this solution to a good quality feasible solution $Y$? Since (11) implies that a feasible solution $Y$ contains exactly a single "1" in each row, in order to satisfy (11) we must insert a "1" into each row that does not contain a "1", and delete all "1‴s but one from each row that contains two or more "1‴s. The algorithm we propose is given below:

**Algorithm AlgUB.**

*STEP* 1: Initialize the matrix $Y$ and the list $L$

For $i = 1, \ldots, I$:

- If the $i$th row of $Y^{(0)}$ has exactly one "1":
  - Set $Y_{i,j} = Y_{i,j}^{(0)}$ for $1 \le j \le J$, i.e. set the $i$th row of $Y$ equal to the $i$th row of $Y^{(0)}$.
- If the $i$th row of $Y^{(0)}$ has either no "1‴s or more than one "1":
  - Set $Y_{i,j} = 0$ for $1 \le j \le J$, i.e. set the $i$th row of $Y$ equal to zeros.

*STEP* 2: Initialize the list $L$

For $i = 1, \ldots, I$:

- If the $i$th row of $Y^{(0)}$ has no "1‴s:
  - For each $j$ such that $X_j = 1$, add the tuple $(i, j, \Delta g(i,j))$ to the list $L$, where $\Delta g(i,j)$ is the amount by which the objective function (10) is increased when the $(i, j)$th entry of $Y$ is changed to a "1".
- If the $i$th row of $Y^{(0)}$ has more than one "1":
  - For each $j$ such that $Y_{i,j}^{(0)} = 1$, add the tuple $(i, j, \Delta g(i,j))$ to the list $L$, where $\Delta g(i,j)$ is the amount

by which the objective function (10) is increased when the $(i,j)$th entry of $Y$ is changed to a "1".

*STEP* 3: Update $Y$ and $L$

Repeat the following steps until the list $L$ is empty:

- Find the tuple $(i, j, \Delta g(i, j))$ in the list having the smallest value of $\Delta g(i, j)$. Let the value of $i$ and $j$ that achieve the minimum be respectively $i_{\min}, j_{\min}$.
- Remove the tuple $(i_{\min}, j_{\min}, \Delta g(i_{\min}, j_{\min}))$ from the list and set $Y_{i_{\min} j_{\min}} = 1$.
- Scan the list and remove any tuple $(i, j, \Delta g(i, j))$ with $i = i_{\min}$.
- Scan the list and recalculate $\Delta g(i, j)$ for any tuple $(i, j, \Delta g(i, j))$ with $j = j_{\min}$.

We note that upon completion of algorithm **AlgUB**, the pair $X$, $Y$ will be a feasible solution. Also, at each step, a "1" is inserted in the location $(i,j)$ that brings about the smallest increase $\Delta g(i, j)$ in the objective function (10). To clarify Step 3, we note that

(i) The reason that tuples $(i, j, \Delta g(i, j))$ with $i = i_{\min}$ are removed in Step 3 is that once $Y_{i_{\min} j_{\min}}$ has been set to one, we already have one "1" in row $i_{\min}$, so we remove any tuples that would place another "1" in this row.

(ii) The reason why $\Delta g(i, j)$ is recalculated for tuples $(i, j, \Delta g(i, j))$ with $j = j_{\min}$ in Step 3 is that the objective function (10) depends on all entries in the $j$th column of $Y$ via the term $\beta_{inv} Z_j^*(Y_{\cdot j})$, and hence when an entry in this column changes, all tuples that would add a "1" to this column must be recalculated.

If the greatest observed lower bound is equal to the smallest observed upper bound within some pre-specified tolerance, we have found an "optimal" solution to (10)–(14). Otherwise, the Lagrange multipliers are updated using subgradient optimization as described in Fisher [13–15] and we repeat algorithms **AlgLB** and **AlgUB** until a feasible solution with the desired tolerance is obtained or the minimum value of the step-size is reached. If, when the Lagrangian procedure terminates, the best known lower bound is equal to the best known upper bound (within some pre-specified tolerance), we have found the optimal solution to the MJIL problem. Otherwise, a branch-and-bound algorithm is used to close the gap, with branching performed on the location variables $X_j$. At each node of the branch-and-bound tree, the distribution center selected for branching is the unopened distribution center with the greatest assigned demand; if all distribution centers in the solution have already been forced open, we branch on an arbitrarily selected unforced distribution center. The variable is first forced to zero and then to one. Branching is done in a depth-first manner. The tree is fathomed at a given node if the lower bound at that node is greater than or equal to the objective value of the best feasible solution found anywhere in the tree to date, or if all distribution centers have been forced open or closed.

## 5. Computational results

In this section, we first explain the design of our computational experiments and then summarize the results. We tested our heuristic for the MJIL problem on a total 1750 randomly generated instances against the Lagrangian relaxation based algorithm used by Diabat et al. [10]. As in Shu [30], the location of the distribution centers and the retailers are uniformly distributed over [0,100] × [0,100]. All transportation costs are assumed to be proportional to the euclidean distance in the plane. Retailers' fixed-order costs, unit-inventory costs, and demands are generated uniformly in [150,250]. For the distribution centers, fixed-order costs are generated in [300,400] and unit-inventory costs are generated uniformly in (0,100]. We ran the three algorithms on the following pairs $(\beta_{trn}, \beta_{inv})$: (0.01,1), (0.01,100), (1,0.01), (1,1), (1,100),

**Table 2**
Parameters for the Lagrangian relaxation algorithm.

| Parameter | Value |
|---|---|
| Maximum number of iterations at each node | 1200 |
| Number of non-improving iterations before halving $\alpha$ | 12 |
| Initial value of $\alpha$ | 2 |
| Minimum value of $\alpha$ | 0.00000001 |
| Minimum gap | 1% |
| Initial value for $\lambda_i$ | $10\overline{\mu} + 10f_i$ |

**Table 3**
Results for $(\beta_{inv}, \beta_{trn}) = (0.01, 1)$.

| Problem size | | Diabat et al. [10] | | Lagrangian heuristics | |
|---|---|---|---|---|---|
| $\|I\|$ | $\|J\|$ | CPU | Gap (%) | CPU | Gap (%) |
| 50 | 10 | 56.6 | 0.196 | 1.7 | 0.359 |
| 50 | 30 | 174.6 | 0.256 | 3.7 | 0.355 |
| 50 | 50 | 285.0 | 0.132 | 4.7 | 0.177 |
| 100 | 50 | 517.7 | 0.142 | 9.8 | 0.238 |
| 100 | 75 | 875.4 | 0.147 | 14.0 | 0.215 |
| 100 | 100 | 1100.8 | 0.126 | 14.3 | 0.160 |
| 150 | 100 | 1683.9 | 0.180 | 15.0 | 0.328 |
| 150 | 125 | 2372.1 | 0.209 | 20.1 | 0.267 |
| 150 | 150 | 2625.6 | 0.137 | 28.5 | 0.216 |
| 250 | 200 | 5124.6 | 0.264 | 32.9 | 0.433 |

**Table 4**
Results for $(\beta_{inv}, \beta_{trn}) = (0.01, 100)$.

| Problem size | | Diabat et al. [10] | | Lagrangian heuristics | |
|---|---|---|---|---|---|
| $\|I\|$ | $\|J\|$ | CPU | Gap (%) | CPU | Gap (%) |
| 50 | 10 | 56.4 | 0.284 | 0.8 | 0.417 |
| 50 | 30 | 176.0 | 0.088 | 2.4 | 0.176 |
| 50 | 50 | 320.2 | 0.156 | 4.2 | 0.200 |
| 100 | 50 | 595.7 | 0.165 | 4.2 | 0.273 |
| 100 | 75 | 829.1 | 0.203 | 7.1 | 0.319 |
| 100 | 100 | 1241.9 | 0.129 | 9.1 | 0.232 |
| 150 | 100 | 1590.7 | 0.177 | 9.2 | 0.303 |
| 150 | 125 | 1895.8 | 0.221 | 11.0 | 0.356 |
| 150 | 150 | 2391.7 | 0.114 | 13.2 | 0.220 |
| 250 | 200 | 5041.2 | 0.093 | 16.1 | 0.163 |

**Table 5**
Results for $(\beta_{inv}, \beta_{trn}) = (1, 0.01)$.

| Problem size | | Diabat et al. [10] | | Lagrangian heuristics | |
|---|---|---|---|---|---|
| $\|I\|$ | $\|J\|$ | CPU | Gap (%) | CPU | Gap (%) |
| 50 | 10 | 61.5 | 0.208 | 1.0 | 0.367 |
| 50 | 30 | 190.6 | 0.264 | 2.5 | 0.412 |
| 50 | 50 | 265.4 | 0.110 | 4.9 | 0.152 |
| 100 | 50 | 530.5 | 0.203 | 4.6 | 0.331 |
| 100 | 75 | 820.3 | 0.256 | 7.1 | 0.359 |
| 100 | 100 | 1078.8 | 0.270 | 9.4 | 0.429 |
| 150 | 100 | 1740.4 | 0.189 | 8.3 | 0.307 |
| 150 | 125 | 2186.8 | 0.154 | 12.0 | 0.209 |
| 150 | 150 | 2848.2 | 0.241 | 12.5 | 0.362 |
| 250 | 200 | 5267.6 | 0.116 | 19.5 | 0.213 |

(100,0.01), and (100,1). The values for these parameters were chosen in this manner to provide a large range of tradeoffs between location costs, transportation costs, and inventory costs. For every pair $(\beta_{trn}, \beta_{inv})$ we ran 25 instances for every problem size.

The parameters that we used for the Lagrangian relaxation procedure are given in Table 2. The notation $\overline{\mu}$ stands for the average demand for all retailers. We terminated our algorithm when the optimality gap was below 1%, or the maximum number of iterations allowed or the minimum value of $\alpha$ (the scalar used to calculate the step-size) occurred. For a more detailed explanation of the Lagrangian relaxation parameters, see Daskin [7]. We coded the algorithms in C++ and ran them on a 2.26 GHz dual processor Dell Precision T7500 workstation with 12 GB of RAM.

Tables 3–9 summarize the results of our computational studies. For Diabat et al. [10], the gap is defined by $(Z_{UB}^D - Z_{LB}^D)/Z_{LB}^D$, where $Z_{UB}^D$ and $Z_{LB}^D$ are respectively the best upper bound and best lower bound obtained by the same algorithm. Whereas for the Lagrangian relaxation heuristic, the value gap is defined by $(Z_{UB}^H - Z_{LB}^D)/Z_{LB}^D$, where $Z_{UB}^H$ represents the best upper bound obtained using our new heuristic. The reason for using the best lower bound obtained by Diabat et al. in calculating the gap for our new heuristic is that algorithm **AlgLB** might give a solution that is not really a lower bound to the MJIL problem and consequently may be higher than the best known upper bound. However, during our computational experience with the 1750 instances, we never observed this.

As can be seen from Tables 3–9, our Lagrangian relaxation heuristic is always able to obtain a solution that is within 0.5% of the solution obtained by Diabat et al. [10], but in much smaller computational times.

**Table 6**
Results for $(\beta_{inv}, \beta_{trn}) = (1, 1)$.

| Problem size | | Diabat et al. [10] | | Lagrangian heuristics | |
|---|---|---|---|---|---|
| $\lvert I \rvert$ | $\lvert J \rvert$ | CPU | Gap (%) | CPU | Gap (%) |
| 50 | 10 | 63.5 | 0.233 | 0.8 | 0.366 |
| 50 | 30 | 168.6 | 0.105 | 3.1 | 0.176 |
| 50 | 50 | 305.2 | 0.087 | 5.2 | 0.160 |
| 100 | 50 | 636.1 | 0.210 | 5.9 | 0.347 |
| 100 | 75 | 815.9 | 0.166 | 8.4 | 0.261 |
| 100 | 100 | 1120.3 | 0.171 | 11.6 | 0.313 |
| 150 | 100 | 1559.0 | 0.156 | 10.8 | 0.309 |
| 150 | 125 | 2165.4 | 0.217 | 12.9 | 0.368 |
| 150 | 150 | 2444.3 | 0.304 | 14.6 | 0.385 |
| 250 | 200 | 5913.8 | 0.168 | 19.8 | 0.243 |

**Table 7**
Results for $(\beta_{inv}, \beta_{trn}) = (1, 100)$.

| Problem size | | Diabat et al. [10] | | Lagrangian heuristics | |
|---|---|---|---|---|---|
| $\lvert I \rvert$ | $\lvert J \rvert$ | CPU | Gap (%) | CPU | Gap (%) |
| 50 | 10 | 61.6 | 0.144 | 1.1 | 0.214 |
| 50 | 30 | 172.1 | 0.159 | 2.5 | 0.236 |
| 50 | 50 | 300.8 | 0.186 | 4.8 | 0.266 |
| 100 | 50 | 566.6 | 0.335 | 6.0 | 0.446 |
| 100 | 75 | 844.6 | 0.321 | 8.9 | 0.417 |
| 100 | 100 | 1236.6 | 0.085 | 10.3 | 0.166 |
| 150 | 100 | 1779.8 | 0.127 | 8.1 | 0.199 |
| 150 | 125 | 2389.8 | 0.305 | 12.6 | 0.398 |
| 150 | 150 | 2782.3 | 0.119 | 15.1 | 0.221 |
| 250 | 200 | 6006.0 | 0.172 | 23.0 | 0.219 |

**Table 8**
Results for $(\beta_{inv}, \beta_{trn}) = (100, 0.01)$.

| Problem size | | Diabat et al. [10] | | Lagrangian heuristics | |
|---|---|---|---|---|---|
| $\lvert I \rvert$ | $\lvert J \rvert$ | CPU | Gap (%) | CPU | Gap (%) |
| 50 | 10 | 57.7 | 0.125 | 0.9 | 0.162 |
| 50 | 30 | 175.2 | 0.286 | 2.9 | 0.392 |
| 50 | 50 | 288.5 | 0.307 | 4.3 | 0.446 |
| 100 | 50 | 576.3 | 0.136 | 4.1 | 0.196 |
| 100 | 75 | 887.4 | 0.190 | 6.0 | 0.325 |
| 100 | 100 | 1033.5 | 0.245 | 10.0 | 0.408 |
| 150 | 100 | 1670.0 | 0.223 | 9.5 | 0.417 |
| 150 | 125 | 2349.2 | 0.243 | 11.2 | 0.413 |
| 150 | 150 | 2782.0 | 0.204 | 13.0 | 0.380 |
| 250 | 200 | 6283.6 | 0.291 | 18.7 | 0.421 |

**Table 9**
Results for $(\beta_{inv}, \beta_{trn}) = (100, 1)$.

| Problem size | | Diabat et al. [10] | | Lagrangian heuristics | |
|---|---|---|---|---|---|
| $\lvert I \rvert$ | $\lvert J \rvert$ | CPU | Gap (%) | CPU | Gap (%) |
| 50 | 10 | 57.7 | 0.319 | 0.9 | 0.404 |
| 50 | 30 | 175.2 | 0.111 | 2.9 | 0.187 |
| 50 | 50 | 288.5 | 0.221 | 4.3 | 0.277 |
| 100 | 50 | 576.3 | 0.156 | 4.1 | 0.284 |
| 100 | 75 | 887.4 | 0.107 | 6.0 | 0.172 |
| 100 | 100 | 1033.5 | 0.197 | 10.0 | 0.289 |
| 150 | 100 | 1670.0 | 0.236 | 9.5 | 0.448 |
| 150 | 125 | 2349.2 | 0.215 | 11.2 | 0.413 |
| 150 | 150 | 2782.0 | 0.115 | 13.0 | 0.209 |
| 250 | 200 | 6283.6 | 0.258 | 18.7 | 0.450 |

## 6. Conclusion and future research

In this paper we studied an integrated supply chain model that considers facility location decisions and inventory decisions simultaneously. The model combines the one-distribution center multi-retailer inventory problem with the uncapacitated fixed-charge location problem. The model aims to determine: (1) the number of distribution centers to establish; (2) their location; (3) the sets of retailers that are assigned to each distribution center; and (4) the size and timing of orders for each facility so as to minimize the sum of inventory, shipping, ordering, and location costs while satisfying end-customer demand.

Due to the success that Lagrangian relaxation has exhibited in tackling several NP-hard supply chain combinatorial optimization problems, we chose to address the MJIL with a Lagrangian relaxation-based heuristic. After decomposing the problem by location, we are able to consider each distribution center location separately. A simple greedy heuristic is implemented to assign retailers to each distribution center, and a lower bound is obtained for the problem. Another algorithm is developed to obtain an upper bound to the problem and if the greatest observed lower bound is identical to the lowest observed upper bound, within a pre-specified tolerance, the optimal solution to the problem is found. Otherwise, the values of the Lagrange multipliers are updated by means of subgradient optimization and the lower and upper bound algorithms are repeated until a feasible solution is reached, that satisfies the given tolerance.

Our computational tests were performed for 1750 problem instances, based on problems of 10 different sizes, and the algorithm was terminated each time when a gap of less than 1% was achieved, or if the maximum number of iterations was reached. Results demonstrate that the proposed Lagrangian relaxation framework is capable of efficiently producing optimal or near-optimal solutions to the problem. The sub-problems are solved heuristically and this means that the lower bound obtained from the constructed algorithm could in fact exceed the best known upper bound. However, this was not observed in any of the problem instances, which proves that our approach is robust and reliable.

The research presented in this paper can be extended in a number of important ways. The structure of the model considered in this paper is such that new constraints or cost components can be added easily to the model. The following are some recommendations for future work and research directions for enhancing the model: (1) The model can be naturally extended to consider multiple products. (2) We have assumed that there is no capacity restriction on the amount of product that can be stored or processed by a facility. We can replace the uncapacitated fixed charge location problem by the capacitated fixed charge location problem and then integrate this with the proposed inventory model. The resulting model would include capacity considerations at the distribution centers. (3) We can relax the single-sourcing restriction to allow a single retailer to be supplied by more than one distribution center. This relaxation is practical in capacitated models or in models with multiple products. (4) Another important extension to our model is to allow lateral shipments between distribution centers. A distribution center may face a demand that exceeds its inventory for a certain product that could be shipped from another distribution center with excess inventory for that product. Lateral shipments are known to reduce costs in practice especially when both distribution centers (the provider and the recipient) are owned by the same firm. Even if these distribution centers belong to different firms, the concept of lateral shipments can still reduce costs because a firm with inventory in excess of demand would generally be willing to sell it at a reduced price. (5) We have assumed direct shipments between the distribution centers and the retailers. An important extension is to incorporate routing decisions. The resulting model would then become a location-inventory-routing model.

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