Decision-Aid Methods Based on Belief Function Theory with Application to Torrent Protection

Simon Carladous, Jean-Marc Tacnet, Jean Dezert, Mireille Batton-Hubert

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Abstract In mountainous areas, decision-makers must find the best solution to protect elements-at-torrential risk. The decision process involves several criteria and is based on imperfect information. Classical Multi-Criteria Decision-Aiding methods (MCDAs) are restricted to precise criteria evaluation for decision-making under a risky environment and suffer of rank reversal problems. To bridge these gaps, several MCDAs have been recently developed within belief function theory framework. The aims of this chapter are to introduce how these methods can be applied in practice and to introduce their general principles. To show their applicability to the real-life problem, we apply them to the Decision-Making Problem (DMP) comprising the comparison of several protective alternatives against torrential floods and selection of the most efficient one. We finally discuss the method improvements to promote their practical implementation.

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15.1 Introduction

Mountainous torrential floods are different from plains' floods because of high flow velocity and high concentration of materials in flowing. Materials come from headwaters, are transported in a channel by debris flows or bedload transport, and finally spread on the alluvial fan. As shown in Fig. 15.1, they put people, buildings, and networks at risk.

Protective systems aim is to reduce damage on elements-at-risk. Therefore, they have specific functions. For instance, the check dam series maintain materials in headwaters whereas sediment traps stop materials before they reach elements-at-risk [1]. In practice, risk managers decide on actions based on several criteria, for example, cost vs. damage reduction. An example of a practical Decision-Making Problem (DMP) is given in Fig. 15.2. The goal is to compare several potential protective actions A_i within a torrential watershed (i) to assign each alternative to a class (or label) as classically done by experts, or (ii) to rank all alternatives according to a preference order, or (iii) to choose the best alternative [2].

Classical Multi-Criteria Decision-Aiding methods (MCDAs) such as Analytic Hierarchy Process (AHP) [3], Technique for Order Preference by Similarity to the Ideal Solution (TOPSIS) [4], and Cost-Benefit Analysis (CBA) [5] help to make decision on such Multi-Criteria Decision-Making (MCDM) problems. While evaluations of criteria in practice are done with different units and scales, imperfect, provided by more or less reliable sources, and made under an epistemically uncertain environment [6], classical MCDAs only consider perfect criteria evaluation, suffer from rank reversal problems, and are limited to decisions under a risky environment.
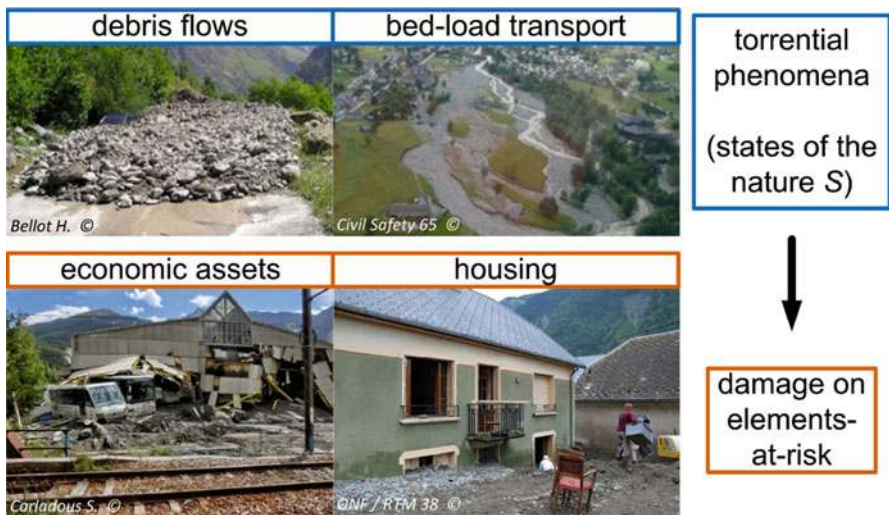

Fig. 15.1 Torrential phenomena and examples of elements-at-risk

To bridge these gaps, several MCDAs were developed within belief function theory framework. As for any MCDA, the first step is to specify the DMP, potential alternatives, decision criteria [2], their scoring scale, and their importance weights. To assign each alternative to a qualitative label (i.e., good, bad, very bad, etc.), Evidential Reasoning for Multi-Criteria Decision Analysis (ER-MCDA) approach extends the AHP by taking into account imperfect evaluation of each criterion provided by several sources. Belief function theory is coupled [7] with fuzzy sets [8] and possibility theories [9]. Belief Function-based Technique for Order Preference by Similarity to the Ideal Solution (BF-TOPSIS) methods are more robust to rank reversal phenomena to rank all alternatives than classical MCDAs [10]. Cautious Ordered Weighted Averaging with Evidential Reasoning (COWA-ER) [11] and Fuzzy COWA-ER (FCOWA-ER) [12] improve initial OWA [13] to help to make decision under an epistemically uncertain environment.

This chapter shows how these new methods can be combined and applied in practice. Therefore, Sect. 15.2 not only recalls basics of MCDM problems and decision-making under uncertainty but also basics of fuzzy set, possibility, and belief function theories. Section 15.3 introduces general principles of ER-MCDA, BF-TOPSIS, and FCOWA-ER methods. They are then applied in Sect. 15.4 to the same DMP introduced in Fig. 15.2. In Sect. 15.5, we finally discuss needed improvements to encourage their practical implementation.
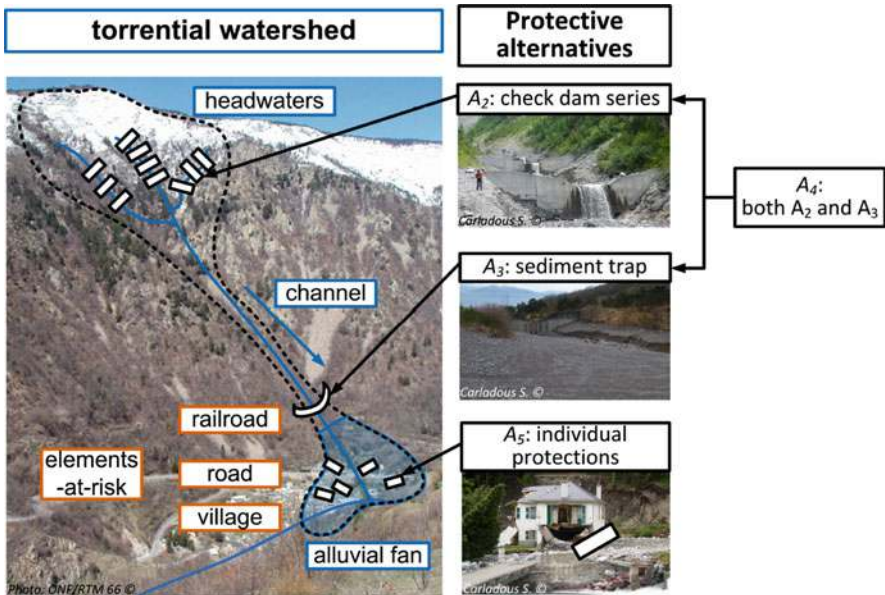

Fig. 15.2 A real-life DMP within a torrential watershed

15.2 Basics of Decision-Making and Imperfect Information

This section provides a formal description of the DMPs, models of representing imperfect information, and decision methods given it.

15.2.1 Formalization of Decision-Making Problems

Any DMP is about comparing M alternatives $A_i \in \mathscr{A}$ and selecting the best one. A decision-maker (DM) faces a MCDM problem if decision depends on several criteria g_j, $j = 1, \ldots, N$. A set $\mathscr{S}$ represents the states of the nature. Since the beginning of the twentieth century, it has been proposed to distinguish decision-making under risk from decision-making under uncertainty, given the DM knowledge on $\mathscr{S}$ [14]. This subsection introduces related formalisms to represent the whole MCDM problem under uncertainty.

15.2.1.1 Multi-criteria Decision-Making Problem

A DM assigns an importance weight ω_j to each criterion g_j, $j = 1, \ldots, N$. Respecting the condition $\sum_{j=1}^{N} \omega_j = 1$, the vector $\mathbf{w} = [\omega_1, \ldots, \omega_j, \ldots, \omega_N]$ represents the DM preference over these criteria. For each g_j, a specific scoring scale X_j is defined. The DM scores each alternative A_i based on each g_j. This score is denoted $x_{ij} \in X_j$. The DM eventually provides the $M \times N$ score matrix $\mathbf{S} = [x_{ij}]$ defined by Eq. (15.1) [15].

$$
\mathbf{S} \triangleq
\begin{array}{c}
 \\
A_1 \\
\vdots \\
A_i \\
\vdots \\
A_M
\end{array}
\begin{array}{c}
g_1, \omega_1 \quad \ldots \quad g_j, \omega_j \quad \ldots \quad g_N, \omega_N \\
\left(
\begin{array}{ccccc}
x_{11} & \ldots & x_{1j} & \ldots & x_{1N} \\
\vdots & & \vdots & & \vdots \\
x_{i1} & \ldots & x_{ij} & \ldots & x_{iN} \\
\vdots & & \vdots & & \vdots \\
x_{M1} & \ldots & x_{Mj} & \ldots & x_{MN}
\end{array}
\right)
\end{array}
\tag{15.1}
$$

15.2.1.2 Decision-Maker Preferences

The DM has preferences not only over criteria but also to compare alternatives according to each criterion. First, AHP helps to establish the N-vector $\mathbf{w}$ which represents the DM preferences over criteria by comparing criteria pairwisely [3]. Second, DM has a preference ordering between all alternatives $A_i \in \mathscr{A}$, given a scoring scale X_j for each g_j. For instance, considering three alternatives A_1, A_2, and A_3 and their scores $x_{1j}, x_{2j}, x_{3j} \in X_j$, the DM preference can be represented by:

- a total pre-order which assumes preference transitivity: if $A_1 \succ A_2$ and $A_2 \succ A_3$, then $A_1 \succ A_3$ [16].
- a partial pre-order which relaxes transitivity assumption [2].

15.2.1.3 From Decision-Making Under Risk to Decision-Making Under Uncertainty

In practice, torrential hazard is generally represented by a finite set of states of the nature $\mathscr{S} = \{S_1, \ldots, S_k, \ldots, S_K\}$, as recalled in Fig. 15.1. Each S_k is commonly referred to as scenario [17]. Given S_k, C_{ik} is the global payoff expected for each alternative A_i. DM provides the $M \times K$ payoff matrix $\mathbf{C} = [C_{ik}]$ defined by Eq. (15.2) [11].

Given $\mathbf{C}$, decision-making depends on the DM knowledge on $\mathscr{S}$ [11]:

- Decision-making under certainty: since only one S_k is known and certain to occur, it consists in choosing the best $A_{i^\star}$ with $i^\star \triangleq \arg\max_i\{C_{ik}\}$.
- Decision-making under risk (or aleatory uncertainty): the true state of the nature is unknown, but one knows all the probabilities $p_k = P(S_k)$. In the context of natural hazards, the expected payoff $E[C_i] = \sum_k p_k \cdot C_{ik}$ is generally computed for each A_i. The best $A_{i^\star}$ is with $i^\star \triangleq \arg\max_i\{E[C_i]\}$.
- Decision-making under ignorance: one assumes no knowledge about the true state of the nature but that it belongs to $\mathscr{S}$. Yager's Ordered Weighted Averaging (OWA) approach [13] can be used to make a decision in this context.
- Decision-making under uncertainty: a belief structure characterizes the knowledge on S. In practice, this is the closest representation of torrential hazard knowledge. Its elicitation by subjective probabilities $p_k = P(S_k)$ is usually used. Thus, decision-making is similar to decision-making under risk [18]. A more interesting approach is the OWA proposed by Yager [13] and improved in [11, 12].

$$
\mathbf{C} \triangleq
\begin{array}{c}
\\ A_1 \\ \vdots \\ A_i \\ \vdots \\ A_M
\end{array}
\begin{array}{c}
\begin{array}{ccccc}
S_1 & \ldots & S_k & \ldots & S_K
\end{array} \\
\left(
\begin{array}{ccccc}
C_{11} & \ldots & C_{1k} & \ldots & C_{1K} \\
\vdots & & \vdots & & \vdots \\
C_{i1} & \ldots & C_{ik} & \ldots & C_{iK} \\
\vdots & & \vdots & & \vdots \\
C_{M1} & \ldots & C_{Mk} & \ldots & C_{MK}
\end{array}
\right)
\end{array}
\tag{15.2}
$$

Formalisms of MCDM problems and decision-making under uncertainty are complementary in representing the DMP. A multi-criteria aggregation based on $\mathbf{S}$ can give each vector $\mathbf{C_k} \triangleq [C_{1k}, \ldots, C_{ik}, \ldots, C_{Mk}]$ of $\mathbf{C}$. For each A_i and g_j, computing the expected payoff $E[C_i]_j$ from $\mathbf{C}$ can provide $x_{ij} = E[C_i]_j$ in $\mathbf{S}$ [6].

15.2.1.4 Several Types of Imperfect Information

Whatever the DMP, decisions depend on the quality of information used to assess scores x_{ij} ($i = 1, \ldots, M$; $j = 1, \ldots, N$), payoffs C_{ik}, and states of the nature S_k ($k = 1, \ldots, K$). There are various types of information imperfection [19]:

- inconsistency, which is related to conflict between sources such as several experts;
- imprecision referring, for example, to interval of numerical values;
- incompleteness, which represents the lack of information while data exist;
- aleatory uncertainty referring to aleatory events;
- epistemic uncertainty, which is linked to the lack of knowledge.

In practice, probabilities are usually used to represent imperfect information. A first criticism is their limit[1] to represent uncertainty, while other formalisms are available: sets for imprecision, fuzzy sets for vagueness [8], possibility distributions, and imprecise probabilities for both uncertainty and imprecision [9, 27]. A second criticism is the use of subjective probabilities [18] both to decide under ignorance [20] and to represent the DM attitude with few information [21]. Belief function theory allows taking into account all types of imperfect information but also to make decisions under ignorance and epistemic uncertainty [22].

15.2.1.5 What Is the Decision-Making Problem About?

A DMP is about comparing the M alternatives A_i gathered in the set $\mathscr{A}$. In practice, three different objectives can be given [2]. For instance, the aim is to compare $M = 4$ potential protective actions A_i within a torrential watershed based on their efficiency:

1. to assign each A_i to a predefined qualitative class (or label) of efficiency such as "high," "medium," "low," and "none" [23];
2. to rank all A_i, $i = 1, \ldots, M$, totally or partially: for instance, $A_3 \succ A_4 \succ A_1 \succ A_2$ is a total order, while $A_3 \sim A_4 \succ A_1 \sim A_2$ is a partial order;
3. to choose the best alternative $A_{i^\star} \in \mathscr{A}$, for instance, A_3.

---

[1]Indeed, the ignorance of a parameter value x belonging to $[a, b]$ is usually modeled by a uniform probability distribution function (pdf) over $[a, b]$, which yields from the probability calculus to a nonuniform pdf of $1/x$ on $[1/b; 1/a]$. This result is not acceptable from the ignorance modeling standpoint because if one has no specific information on x, we cannot get more information on $1/x$ but that $1/x$ belongs to $[1/b; 1/a]$. Therefore the uniform pdf often used to model ignorance in probability theory is problematic.

15.2.2 Imperfect Information: From Representation to Decision-Making

This subsection details the three main steps to take into account imperfect information: (1) representation, (2) combination and propagation, and (3) decision [24].

15.2.2.1 Representation of Imperfect Information Provided by a Source

- **Fuzzy set theory** was developed to represent linguistic assessment of fuzziness [8]. Given individual elements x of the universe of discourse X, the membership function $\mu_\theta(x) \in [0, 1]$ associates each $x \in X$ to the fuzzy set θ with the grade of membership $\mu_\theta(x)$. As shown in Fig. 15.3a, a simple way to represent a membership function is to use a trapezoidal membership function defined by the quadruplet $\{a, b, c, d\}$ in Eq. (15.3) [25]: $[a, d]$ is the fuzzy set support denoted by $supp_\theta$, while $[b, c]$ is its core $core_\theta$. Given X, $\bar{\theta}$ is the complement fuzzy set of θ defined by Eq. (15.4) (Fig. 15.3a), and a mapping model [19] is a set Θ of n fuzzy sets θ_e, for $e = 1, \ldots, n$ (Fig. 15.3b). Given two fuzzy sets θ_1 and θ_2, the membership function $\mu_{\theta_1 \cup \theta_2}$ defined by Eq. (15.5) represents their union (Fig. 15.3c), while their intersection $\mu_{\theta_1 \cap \theta_2}$ is defined by Eq. (15.6) [8].

$$
\begin{cases}
0 & \text{if } x \notin supp_\theta \\
\frac{x-a}{b-a} & \text{if } x \in [a, b] \\
1 & \text{if } x \in core_\theta \\
\frac{x-d}{c-d} & \text{if } x \in [c, d]
\end{cases} \tag{15.3}
$$

$$
\mu_{\bar{\theta}}(x) \triangleq 1 - \mu_\theta(x), x \in X \tag{15.4}
$$

$$
\mu_{\theta_1 \cup \theta_2}(x) \triangleq \max_{x \in X}(\mu_{\theta_1}(x), \mu_{\theta_2}(x)) \tag{15.5}
$$

$$
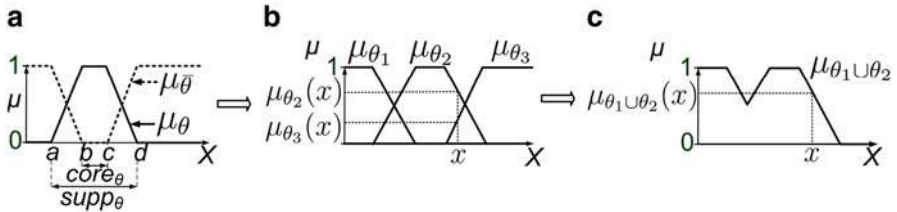\mu_{\theta_1 \cap \theta_2}(x) \triangleq \min_{x \in X}(\mu_{\theta_1}(x), \mu_{\theta_2}(x)) \tag{15.6}
$$

Fig. 15.3 (**a**) Fuzzy set Θ and its complement $\bar{\Theta}$; (**b**) mapping model ($n = 3$); (**c**) union
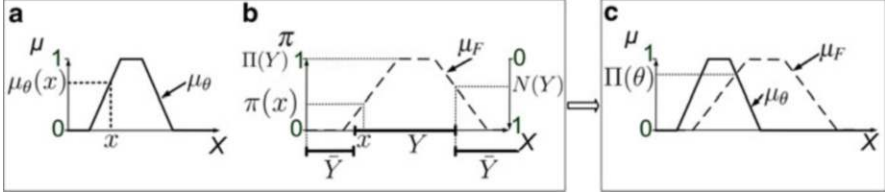
Fig. 15.4 (**a**) Fuzzy set Θ; (**b**) possibility distribution $\pi = \mu_F$; (**c**) possibility measure of Θ given π

- Zadeh, Dubois, and Prade then developed the *possibility theory* in the fuzzy logic framework [9, 27]. Considering the fuzzy set F of possible values of $x \in X$, the possibility distribution π is given by $\mu_F(x) \triangleq \pi(x) \in [0, 1]$ [26]. Given Y a subset of X and $\bar{Y}$ its complement, the possibility and necessity measures are $\Pi(Y) \triangleq \sup_{x \in Y} \pi(x)$ [9] and $N(Y) \triangleq 1 - \Pi(\bar{Y})$, $\forall Y, \bar{Y} \subseteq X$ [27], as shown in Fig. 15.4b. $\Pi(Y)$ and $N(Y)$ are considered as the upper and lower bounds of the probability $P(Y)$. Given X, the membership function μ_θ, and a possibility distribution μ_F (Fig. 15.4c), the possibility measure of θ denoted by $\Pi(\theta)$ is defined by Eq. (15.7) [9].

$$\Pi(\theta) \triangleq \sup_{x \in X} \mu_{\theta \cap F}(x) \tag{15.7}$$

- In the meantime, Shafer introduced the *belief function theory*, also called Dempster-Shafer Theory (DST) [22]. The Frame of Discernment (FoD) is a finite set $\Theta = \{\theta_1, \ldots, \theta_e, \ldots, \theta_n\}$, with $n > 1$, which gathers the potential answers of the DMP under concern. In DST, all FoD elements are assumed exhaustive and mutually exclusive. The power set of Θ denoted by 2^Θ is the set of all subsets of Θ, the empty set $\emptyset$ included. The complement of a subset $A \in 2^\Theta$ is denoted $\bar{A}$. Its cardinality is $|A|$. A source (or body) of evidence is characterized by a basic belief assignment (BBA) $m^\Theta(\cdot)$, which is a mapping $m^\Theta(\cdot) : 2^\Theta \to [0, 1]$ that satisfies $m^\Theta(\emptyset) = 0$, and $\forall A \neq \emptyset \in 2^\Theta$ the condition $\sum_{A \subseteq \Theta} m^\Theta(A) = 1$. The vacuous BBA models the full ignorance of the source of evidence. If $m^\Theta(A) > 0$, A is a focal element of $m^\Theta(\cdot)$. m_A^Θ denotes the categorical BBA which focuses on $A \neq \emptyset$. More precisely, $m_A^\Theta(A) = 1$ and $m_A^\Theta(Y) = 0$ for any $Y \neq A$. Focal elements of a Bayesian BBA are only singletons on 2^Θ. Given the FoD Θ for final decision, it is possible to represent imperfect evaluation of the score x_{ij}, for each alternative A_i according to each criterion g_j, through the BBA $m_{ij}^\Theta(\cdot)$ providing the $M \times N$ BBA matrix $\mathbf{M}^\Theta = [m_{ij}^\Theta(\cdot)]$. Thus, it must be compared with the $M \times N$ score matrix $\mathbf{S} = [x_{ij}]$, defined by Eq. (15.1). Given $m^\Theta(\cdot)$, belief and plausibility functions are, respectively, defined by:

$$\text{Bel}^{\Theta}(A) \triangleq \sum_{Y \subseteq A | Y \in 2^{\Theta}} m^{\Theta}(Y) \tag{15.8}$$

$$\text{Pl}^{\Theta}(A) \triangleq \sum_{Y \cap A \neq \emptyset | Y \in 2^{\Theta}} m^{\Theta}(Y) \tag{15.9}$$

Considering that the universe of discourse X is the FoD Θ, the plausibility measure $\text{Pl}^{\Theta=X}(A)$ is a possibility measure $\Pi(A)$, $\forall A \subseteq \Theta = X$ [26]. $\text{Bel}^{\Theta}(A)$ and $\text{Pl}^{\Theta}(A)$ are interpreted as lower and upper bounds of the unknown probability $\text{P}^{\Theta}(A)$. The interval $\text{BI}^{\Theta}(A) \triangleq [\text{Bel}^{\Theta}(A), \text{Pl}^{\Theta}(A)]$ is its belief interval. Its length $\text{Pl}^{\Theta}(A) - \text{Bel}^{\Theta}(A)$ characterizes the uncertainty, also called ambiguity, on A [28].

Given $m^{\Theta}(\cdot)$, several transformations help to approximate probability function $\text{P}^{\Theta}(\cdot)$: Smets' pignistic transformation provides $\text{BetP}^{\Theta}(\cdot)$ [29], DSmP transformation gives $\text{DSmP}_{\epsilon}^{\Theta}(\cdot)$ where $\epsilon \geq 0$ is a tuning parameter [30] (Vol. 3), and others.

Shafer's exhaustivity assumption means that the FoD is considered as a "closed world" (*c.w.*). In some practical problems, this assumption is too strict and it is more convenient to consider the original FoD as an "open world" (*o.w.*).

1. In Smets' Transferable Belief Model (TBM) [31], $\Theta^{o.w.} \triangleq \{\theta_1, \ldots, \theta_q\}$ and $\emptyset = \bar{\Theta}^{o.w.}$. One has $\sum_{A \in 2^{\Theta}} m(A) = 1$, and one allows $m(\emptyset) \geqslant 0$.
2. In Yager's approach [32], the open world is closed by an hedge element θ^c, so that $\Theta^{c.w.} \triangleq \Theta^{o.w.} \cup \{\theta^c\}$. The cardinality $|\theta^c|$ is not known.

Shafer's mutual exclusivity assumption can be also too strict. Dezert-Smarandache Theory (DSmT) framework modifies DST to relax this assumption and proposes new techniques to combine the sources of evidence and to make a decision [30].

15.2.2.2 Combining Information Provided by Several Sources of Evidence

First of all, the source reliability and its importance must be clearly distinguished. Reliability is the source objective ability to give the correct solution of the DMP [33]. For each source s_q, $q = 1, \ldots, Q$, it is represented by a reliability discounting factor $\alpha_q \in [0, 1]$ [34]. Given the initial BBA $m_q^{\Theta}(\cdot)$ provided by s_q, Shafer's discounting method defined by Eq. (15.10) is generally used to provide the discounted mass $m_{\alpha_q}^{\Theta}(A)$ [22].

$$m_{\alpha_q}^{\Theta}(A) \triangleq \begin{cases} \alpha_q \cdot m_q^{\Theta}(A) & \text{if} A \in 2^{\Theta} \neq \Theta \\ \alpha_q \cdot m_q^{\Theta}(A) + (1 - \alpha_q) & \text{if} A = \Theta \end{cases} \tag{15.10}$$

Importance is the subjective weight granted to the source by DM [33]. In a MCDM problem, each criterion can be considered as a source represented by a BBA $m_j^{\Theta}(\cdot)$. Each weight ω_j is the importance discounting factor used to provide the discounted BBA[2] $m_{\omega_j}^{\Theta}(\cdot)$.

$$
m_{\omega_j}^{\Theta}(A) \triangleq \begin{cases} \omega_j \cdot m_j^{\Theta}(A) & \text{if} A \in 2^{\Theta} \neq \emptyset \\ \omega_j \cdot m_j^{\Theta}(A) + (1 - \omega_j) & \text{if} A = \emptyset \end{cases} \tag{15.11}
$$

Once BBAs have been discounted, the combination of distinct sources of evidence is denoted by $\oplus$ to provide the combined BBA $m_{\oplus}^{\Theta}(A)$, $A \subset \Theta$. The largely used initial Dempster's rule (*DS*) [22] has been subject to strong debates in fusion community, showing it does not behave well in high conflicting case [35] but also in low conflicting cases [36].

As a consequence, since the 1990s, many alternatives have been proposed to combine belief functions more or less efficiently. The Proportional Conflict Redistribution (PCR) rules have been developed in DSmT [30] (Vol. 3) to palliate disadvantages of the classical Dempster's fusion rule [37]. PCR rule n° 6 (PCR6) defined by Eq. (15.12) for combining two sources of evidence ($K = 2$) $m_1^{\Theta}(\cdot)$ and $m_2^{\Theta}(\cdot)$ is also consistent for more than two bodies of evidence ($K > 2$) [38].

$$
m_{\text{PCR6}}^{\Theta}(A) \triangleq \sum_{\substack{X_1, X_2 \in 2^{\Theta} \\ X_1 \cap X_2 = A}} m_1^{\Theta}(X_1) \cdot m_2^{\Theta}(X_2)
$$
$$
+ \sum_{\substack{Y \in 2^{\Theta} \setminus \{A\} \\ A \cap Y = \emptyset}} \left[\frac{m_1^{\Theta}(A)^2 \cdot m_2^{\Theta}(Y)}{m_1^{\Theta}(A) + m_2^{\Theta}(Y)} + \frac{m_2^{\Theta}(A)^2 \cdot m_1^{\Theta}(Y)}{m_2^{\Theta}(A) + m_1^{\Theta}(Y)} \right] \tag{15.12}
$$

Combination by PCR6 fusion rule of the N importance discounted BBAs $m_j^{\Theta}(\cdot)$ defined by Eq. (15.11) provides a BBA denoted $m_{\text{PCR6}_{\emptyset}}^{\Theta}(\cdot)$ with $m_{\text{PCR6}_{\emptyset}}^{\Theta}(\emptyset) > 0$. Then, we commit zero to the mass of the empty set, and we normalize this BBA to get a proper normalized BBA $m_{\text{PCR6}}^{\Theta}(\cdot)$ with $m_{\text{PCR6}}^{\Theta}(\emptyset) = 0$; see [33] for details.

15.2.2.3 Decision-Making Given a Combined Belief Mass

Given a BBA $m^{\Theta}(\cdot)$, choosing a singleton $\hat{\theta} \in \Theta$ or a subset $\hat{A} \subseteq \Theta$ is the decision issue. In general, it consists in choosing $\hat{\theta} = \theta_{e^\star}$, $e = 1, \ldots, n$ with $e^\star \triangleq \arg\max_e C(\theta_e)$, where $C(\theta_e)$ is a decision-making criterion chosen according

---

[2]For a technical reason, one allows to commit some mass on the empty set in this discounting. This is not a problem because the final fusion result will be normalized.

to the DM attitude: belief for a pessimistic attitude, plausibility for an optimistic one, one of the probabilistic transformations for an attitude of compromise.

In general, the DM attitude is not well known in DST. Moreover, in some practical cases, taking into account non-singletons $A \subseteq \Theta$ is needed to decide. For these cases, the minimum of any strict distance metric $d(m^\Theta, m_A^\Theta)$ between $m^\Theta(\cdot)$ and the categorical BBA $m_A^\Theta(\cdot)$ can be used in Eq. (15.13) [39]. If only singletons of 2^Θ are accepted, decision is defined by Eq. (15.14).

$$\hat{A} = \arg \min_{A \in 2^\Theta} d_{\mathrm{BI}}(m^\Theta, m_A^\Theta) \tag{15.13}$$

$$\hat{\theta} = \arg \min_{\theta_e \in \Theta} d_{\mathrm{BI}}(m^\Theta, m_{\theta_e}^\Theta) \tag{15.14}$$

Among the few true distance metrics[3] between two BBAs $m_1^\Theta(\cdot)$ and $m_2^\Theta(\cdot)$, the belief interval-based Euclidean $d_{BI}(m_1^\Theta, m_2^\Theta) \in [0, 1]$ is based on Wasserstein's distance [40] and provides reasonable results [41].

The quality indicator $q(\hat{A})$ defined by Eq. (15.15) evaluates how good the decision $\hat{A}$ is with respect to other focal elements: the higher $q(\hat{A})$ is, the more confident DM should be in its decision $\hat{A}$. If only singletons of 2^Θ are accepted, $q(\hat{\theta})$ is defined by Eq. (15.16).

$$q(\hat{A}) \triangleq 1 - \frac{d_{\mathrm{BI}}(m^\Theta, m_A^\Theta)}{\sum_{A \in 2^\Theta \setminus \{\emptyset\}} d_{\mathrm{BI}}(m^\Theta, m_A^\Theta)} \tag{15.15}$$

$$q(\hat{\theta}) \triangleq 1 - \frac{d_{\mathrm{BI}}(m^\Theta, m_{\theta_e}^\Theta)}{\sum_{e=1}^{n} d_{\mathrm{BI}}(m^\Theta, m_{\theta_e}^\Theta)} \tag{15.16}$$

15.3 Belief Function-Based Decision-Aiding Methods

Classical Decision-Aiding Methods (DAMs) have some limitations: (i) classical MCDAs do not consider imperfect evaluations of criteria, (ii) ranking can be affected by rank reversal problems [42, 43], and (iii) probability framework is limited by an epistemic uncertainty affecting the knowledge on the states of the nature $\mathscr{S}$ [20, 21]. This section introduces new belief function-based DAMs which help to overcome these three limitations using (i) Evidential Reasoning for Multi-Criteria Decision Analysis (ER-MCDA) [44], (ii) Belief Function-based Technique for Order Preference by Similarity to Ideal Solution (BF-TOPSIS) methods [39], and (iii) Fuzzy Cautious Ordered Weighted Averaging with Evidential Reasoning (FCOWA-ER) [12], respectively.

---

[3]For any BBAs x, y, z defined on 2^Θ, a true distance metric $d(x, y)$ satisfies the properties of non-negativity ($d(x, y) \geq 0$), non-degeneracy ($d(x, y) = 0 \Leftrightarrow x = y$), symmetry ($d(x, y) = d(y, x)$), and triangle inequality ($d(x, y) + d(y, z) \geq d(x, z)$).

15.3.1 ER-MCDA: Multi-criteria Assignment Given Imperfect Scores

As detailed in Fig. 15.5, ER-MCDA methodology [44] is an extension of the Analytic Hierarchy Process (AHP) [3]. Given the FoD for decision Θ (step 1), it associates fuzzy logic framework and belief function theory to represent imperfect evaluation of the score of each alternative A_i, $i = 1, \ldots, M$ based on each criterion g_j, $j = 1, \ldots, N$, potentially provided by several sources s_q, $q = 1, \ldots, Q$, through a BBA $m^{\Theta}_{q,ij}(\cdot)$ (step 2). A second improvement is the combination of BBAs taking into account reliability α_q of each source and importance ω_j of each criterion g_j (step 3). It finally helps to assign each A_i to the element $\hat{\theta}(A_i)$ (step 4).

15.3.1.1 ER-MCDA-Step 1: DMP Formalization

The scoring scale X_j is specified for each criterion g_j, $j = 1, \ldots, N$. The FoD for decision Θ is also defined: for instance, we consider four qualitative labels (or classes) θ_e of efficiency with $\Theta = \{\theta_1 = \text{high}, \theta_2 = \text{medium}, \theta_3 = \text{low}, \theta_4 = \text{none}\}$.

15.3.1.2 ER-MCDA-Step 2: BBA $m^{\Theta}_{q,ij}(\cdot)$ Construction

For each g_j, $j = 1, \ldots, N$, the mapping model is first provided (Fig. 15.3): n fuzzy sets μ_{j,θ_e} represent a partial pre-order of the DM preference for X_j [44]. Then, for each alternative A_i, $i = 1, \ldots, M$, each source (e.g., an expert) s_q provides its imprecise and uncertain evaluation of $x_{ij} \in X_j$ through a possibility distribution $\pi_{q,ij}$ (Fig. 15.4).

Given these elements, the mapping process consists in transforming each $\pi_{q,ij}$ into a BBA $m^{\Theta}_{q,ij}(\cdot)$ using the g_j mapping model. The initial mapping process [19] was based on a geometric transformation and was restricted to provide only Bayesian BBAs. A new mapping model was recently developed to provide general BBAs [45, 46], for each A_i and each g_j:

1. since fuzzy sets are given for an open world, Yager's hedged element θ^c [32] is used to provide membership functions in an hedged world (c.w.), and all membership functions $\mu_{j,X}$, $X \neq \emptyset \in \Theta$ are built applying Eq. (15.5);
2. all membership functions $\mu_{j,\bar{X}}$, $X \neq \emptyset \in \Theta$ are built applying Eq. (15.4);
3. given the possibility distribution $\pi_{q,ij}$, Eq. (15.7) gives the possibility measures $\Pi_{q,ij}(\bar{X})$ corresponding to the plausibility measure $\text{Pl}^{\Theta}_{q,ij}(\bar{X})$;
4. the belief function $\text{Bel}^{\Theta}_{q,ij}(\cdot)$ is directly obtained such as the BBA $m^{\Theta}_{q,ij}(\cdot)$.
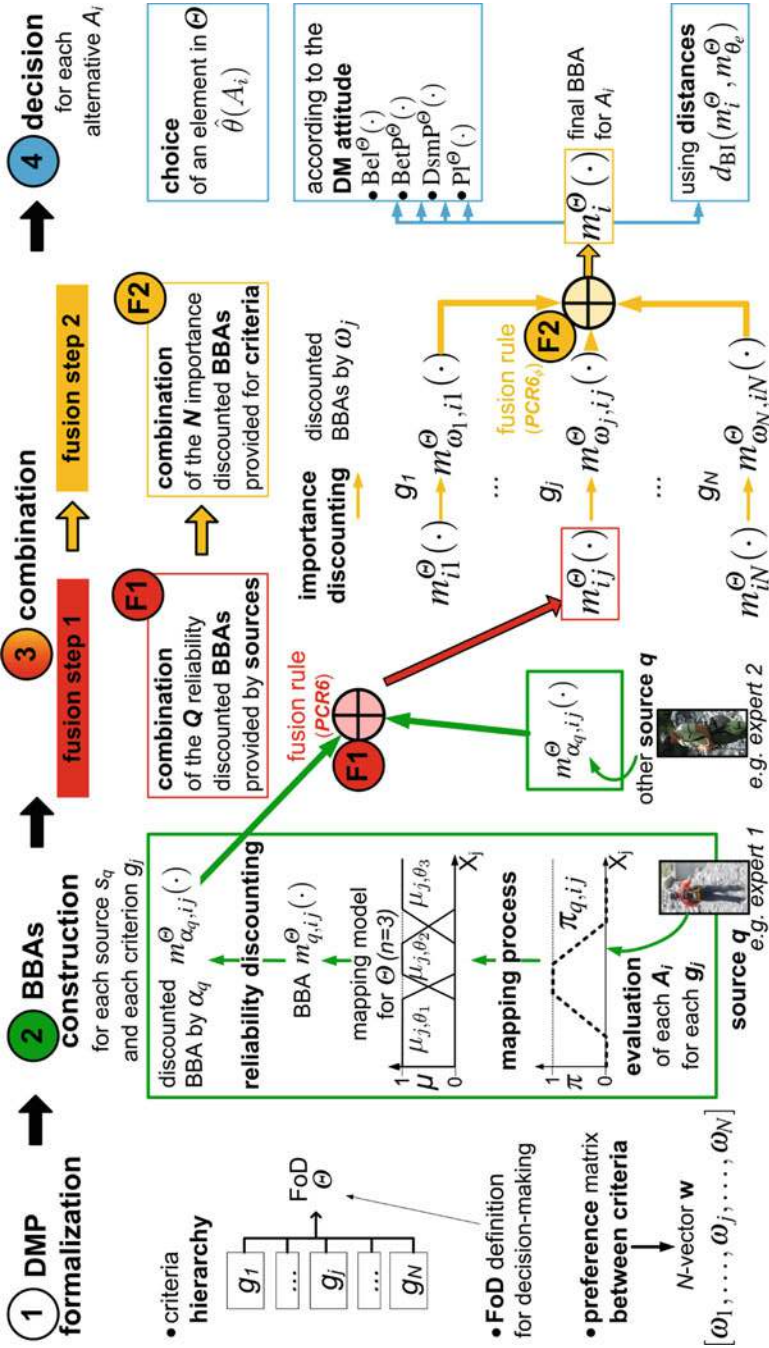
Fig. 15.5 The four steps of ER-MCDA

15.3.1.3 ER-MCDA-Step 3: Combination of Two BBAs

For each A_i and each g_j, fusion 1 combines BBAs provided by Q sources s_q, taking into account reliability factor α_q and using PCR6 fusion rule. For each A_i, fusion 2 combines BBAs provided for N criteria g_j, taking into account importance factor ω_j and using PCR6$_\emptyset$ fusion rule. Final BBA $m_i^\Theta(\cdot)$ is obtained.

15.3.1.4 ER-MCDA-Step 4: Making Decision for Each A_i

Given Θ and $m_i^\Theta(\cdot)$, DM must assign each A_i to labels by choosing $\hat{\theta}(A_i)$. Therefore, DM can decide according to a pessimistic attitude (max of belief), an optimistic one (max of plausibility), or an attitude of compromise (max of subjective probability).

For the latter, whatever the probability transformation, the cardinality $|\theta^c|$ must be known. Therefore, a strong hypothesis is to consider $|\theta^c| = 1$ which can be theoretically discussed. Using Eq. (15.14) to decide through the minimal of belief interval distance does not involve any hypothesis on it.

15.3.2 BF-TOPSIS: A More Robust Multi-criteria Ranking

Given the score matrix $\mathbf{S}$ defined by Eq. (15.1), the classical MCDAs such as the AHP [3], Technique for Order Preference by Similarity to the Ideal Solution (TOPSIS) [4], or Estimator Ranking Vector (ERV) [47] are limited by rank reversal problems [42, 43]. As detailed in Fig. 15.6, the four new BF-TOPSIS methods [10] are inspired by the ERV to avoid a normalization step and by TOPSIS to compare each A_i with an ideal best and an ideal worst solutions. They are based on a common preliminary step and have an increasing computation complexity and robustness to rank reversal problems [6, 10].

15.3.2.1 Preliminary Step: DMP Formalization and BBA $m_{ij}^{\mathscr{A}}(\cdot)$ Construction

A DMP is about ranking all alternatives A_i and choosing the best one $A_{i^\star} \in \mathscr{A}$: the FoD Θ for decision is the set of alternatives $\mathscr{A}$. Given the score matrix $\mathbf{S}$ (see Fig. 15.6), this common step consists of constructing the $M \times N$ BBA matrix $\mathbf{M}^{\mathscr{A}} = [m_{ij}^{\mathscr{A}}(\cdot)]$.

For each A_i and g_j, the positive support $\text{Sup}_j(A_i) \triangleq \sum_{k \in \{1,\dots,M\}|x_{kj} \leq x_{ij}} |x_{ij} - x_{kj}|$ and the negative one $\text{Inf}_j(A_i) \triangleq - \sum_{k \in \{1,\dots,M\}|x_{kj} \geq x_{ij}} |x_{ij} - x_{kj}|$, respectively, measure how much A_i is better and worse than other alternatives according to g_j. Given $A_{\max}^j \triangleq \max_i \text{Sup}_j(A_i)$ and $A_{\min}^j \triangleq \min_i \text{Inf}_j(A_i)$, each $m_{ij}^{\mathscr{A}}(\cdot)$ is defined by:
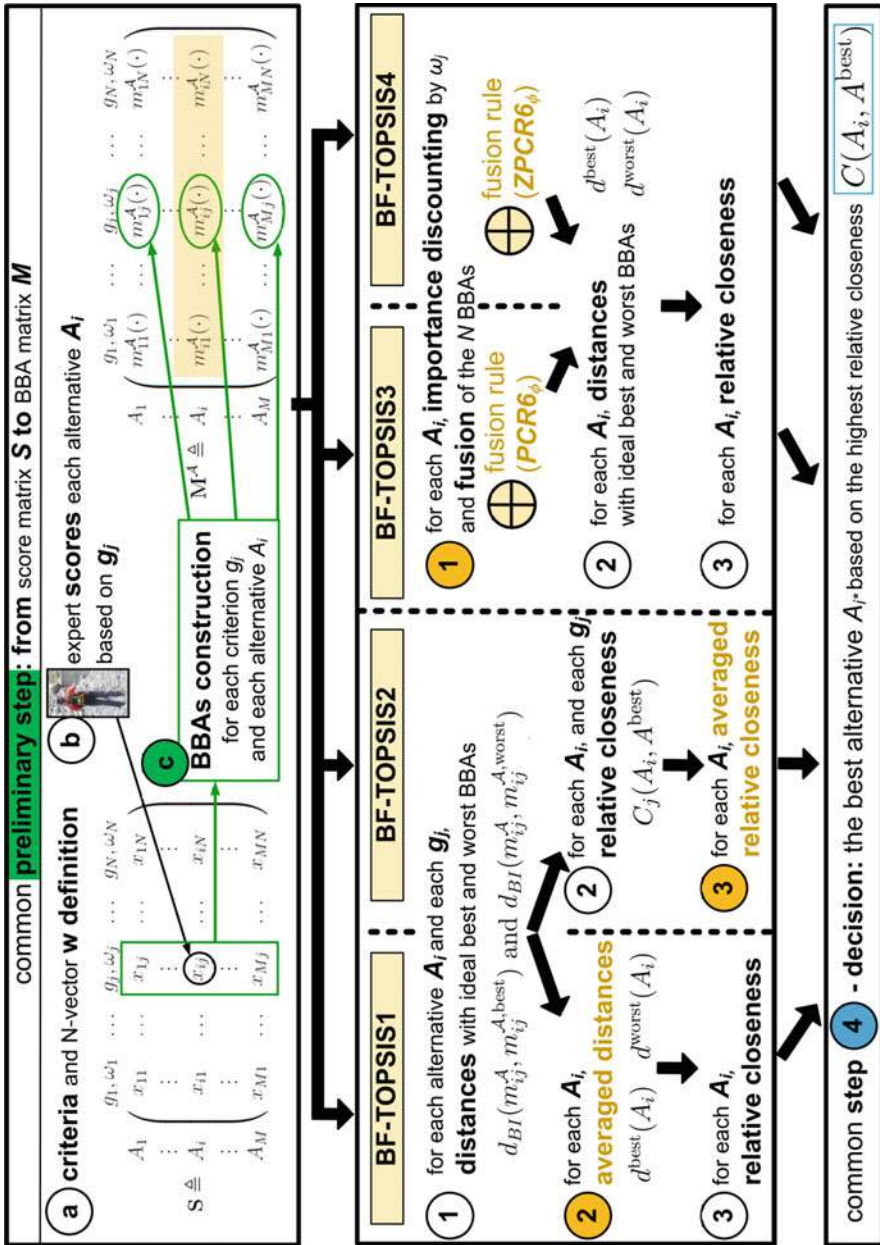
Fig. 15.6 Main steps of the four BF-TOPSIS methods

$$m_{ij}^{\mathscr{A}}(A_i) \triangleq \begin{cases} \frac{\text{Sup}_j(A_i)}{A_{\max}^j} & \text{if} A_{\max}^j \neq 0 \\ 0 & \text{if} A_{\max}^j = 0 \end{cases} \qquad (15.17)$$

$$m_{ij}^{\mathscr{A}}(\bar{A}_i) \triangleq \begin{cases} \frac{\text{Inf}_j(A_i)}{A_{\min}^j} & \text{if} A_{\min}^j \neq 0 \\ 0 & \text{if} A_{\min}^j = 0 \end{cases} \qquad (15.18)$$

$$m_{ij}^{\mathscr{A}}(A_i \cup \bar{A}_i) \triangleq m_{ij}^{\mathscr{A}}(\Theta) \triangleq 1 - (\text{Bel}_{ij}^{\mathscr{A}}(\bar{A}_i) + \text{Bel}_{ij}^{\mathscr{A}}(A_i)) \qquad (15.19)$$

The four BF-TOPSIS methods differ from each other in the way they process the matrix $\mathbf{M}^{\mathscr{A}}$. All of them compute the relative closeness of each alternative A_i with an ideal best solution A^{best} denoted by $C(A_i, A^{\text{best}})$. The preference ordering of all alternatives is built given the following criterion: the higher it is, the better A_i is. An extension of BF-TOPSIS methods for dealing with imprecise scores is proposed in [48].

15.3.2.2 BF-TOPSIS1

(1) For each A_i and g_j, the ideal best and worst BBAs are defined by $m_{ij}^{\text{best}}(A_i) \triangleq 1$ and $m_{ij}^{\text{worst}}(\bar{A}_i) \triangleq 1$ which are used to compute the distances $d_{BI}(m_{ij}^{\mathscr{A}}, m_{ij}^{\mathscr{A},\text{best}})$ and $d_{BI}(m_{ij}^{\mathscr{A}}, m_{ij}^{\mathscr{A},\text{worst}})$.

(2) The respective averaged distances $d^{\text{best}}(A_i)$ and $d^{\text{worst}}(A_i)$ are computed by weighting previous distances by importance weights ω_j of criteria g_j.

(3) For each A_i, the relative closeness is computed by:

$$C\left(A_i, A^{\text{best}}\right) \triangleq \frac{d^{\text{worst}}(A_i)}{d^{\text{worst}}(A_i) + d^{\text{best}}(A_i)} \qquad (15.20)$$

15.3.2.3 BF-TOPSIS2

(1) This step is the same as for BF-TOPSIS1. (2) For each A_i and g_j, the relative closeness $C_j(A_i, A^{\text{best}})$ is computed. (3) The averaged relative closeness $C(A_i, A^{\text{best}})$ is computed by weighting $C_j(A_i, A^{\text{best}})$ by importance weights ω_j of criteria g_j.

15.3.2.4 BF-TOPSIS3

(1) For each A_i, the N BBAs $m_{ij}^{\mathscr{A}}(\cdot)$ are combined through PCR6 fusion rule to give $m_i^{\mathscr{A}}(\cdot)$ taking into account the importance factor ω_j of each criterion g_j [33]. (2) For

each A_i, the ideal best and worst BBAs allow to give $d^{\text{best}}(A_i) = d_{BI}(m_i^{\mathscr{A}}, m_i^{\mathscr{A},\text{best}})$ and $d^{\text{worst}}(A_i) = d_{BI}(m_i^{\mathscr{A}}, m_i^{\mathscr{A},\text{worst}})$. (3) This step is the same as for BF-TOPSIS1.

15.3.2.5 BF-TOPSIS4

This method differs from BF-TOPSIS3 only by the choice of the ZPCR6 fusion rule [49] instead of PCR6 rule of combination.

15.3.3 FCOWA-ER: Choice Under Epistemic Uncertainty

Given the payoff matrix **C** defined by Eq. (15.2), the DMP requires choosing the best alternative $A_{i\star} \in \mathscr{A}$. COWA-ER has been proposed [11] for such decision-making given uncertain knowledge on $\mathscr{S}$. It mixes cautiously the principle of Yager's Ordered Weighted Averaging (OWA) approach based on belief function theory [13] with fusion rules, notably the PCR6 one [30]. As detailed in Fig. 15.7, FCOWA-ER [12] is a modified version of COWA-ER using fuzzy sets which improves performances of COWA-ER and reduces its computational burden.
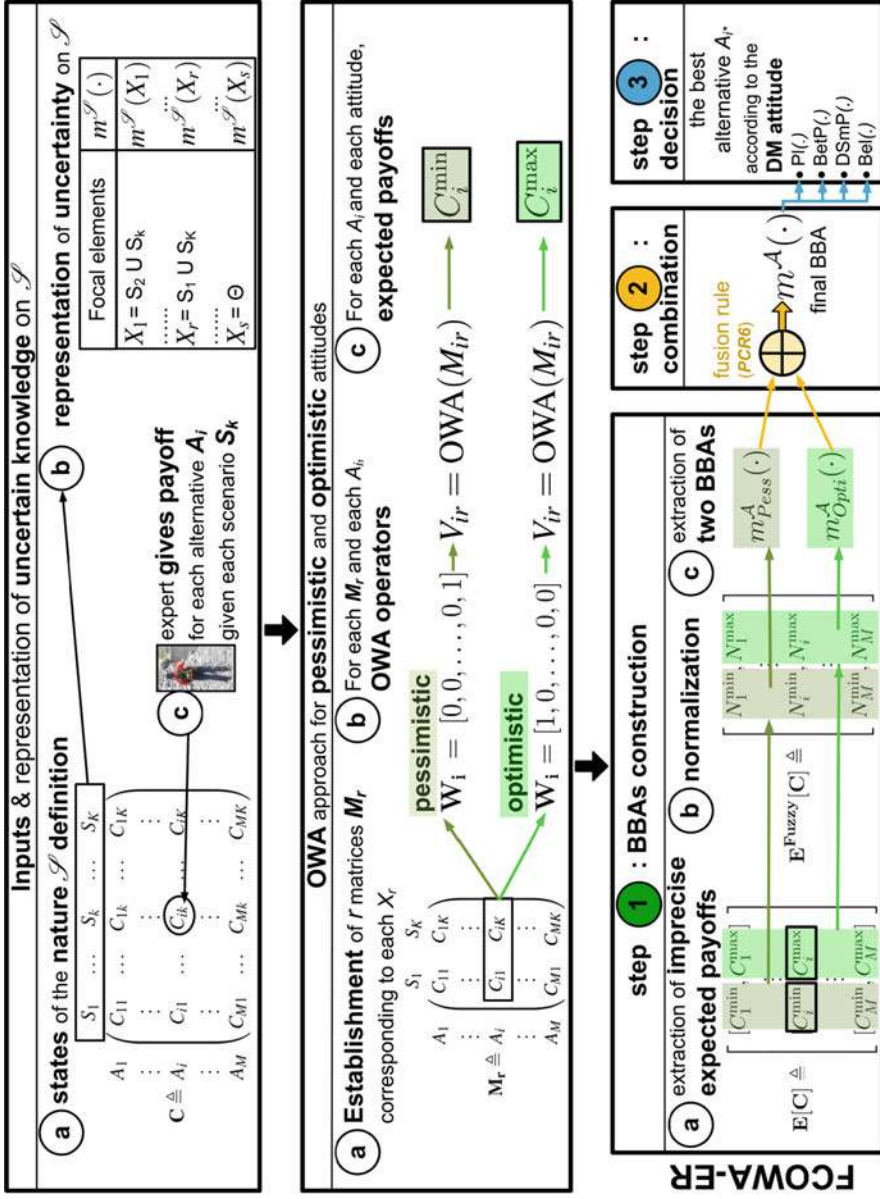
15.3.3.1   From the OWA Approach...

Under ignorance, Yager uses the OWA operator as a weighted average of ordered values of a variable defined by Eq. (15.21). For each A_i, $i = 1, \ldots, M$, it consists in choosing a normalized set of weighting factors $\mathbf{W_i} = [w_{i1}, \ldots, w_{ik}, \ldots, w_{iK}]$, where $w_{ik} \in [0, 1]$, $\sum_k w_{ik} = 1$, and $\mathbf{W_i}$ depends on the DM attitude: $\mathbf{W_i} = [0, 0, \ldots, 0, 1]$ represents the pessimistic attitude, while $\mathbf{W_i} = [1, 0, \ldots, 0, 0]$ is used for the optimistic one. The OWA value V_i is computed for the collection of payoffs $C_{i1}, C_{i2}, \ldots, C_{iK}$, with b_{ik} as the kth largest element in it. The best $A_{i\star}$ is chosen with $i^{\star} \triangleq \arg\max_i\{V_i\}$.

Under epistemic uncertainty, considering the states of the nature $\mathscr{S}$ as the FoD, Yager represents the DM belief structure by a BBA $m^{\mathscr{S}}(\cdot) : 2^{\mathscr{S}} \to [0, 1]$, which is characterized by the s focal elements $X_r \in 2^{\mathscr{S}}$. For each alternative A_i, restricting the states of the nature to $S_k \in X_r$, one has $\mathbf{M_{ir}} \triangleq \{C_{ik}|S_k \in X_r\}$, $r = 1, \ldots, s$. For each A_i, each X_r, and some DM attitude chosen a priori, the OWA value $V_{ir} = \text{OWA}(\mathbf{M_{ir}})$ is computed. The derivation of a generalized expected value C_i of payoff is defined by Eq. (15.22). The best $A_{i\star}$ is thus chosen with $i^{\star} \triangleq \arg\max_i\{C_i\}$.

$$V_i \triangleq \text{OWA}(C_{i1}, C_{i2}, \ldots, C_{iK}) = \sum_k w_{ik} \cdot b_{ik} \qquad (15.21)$$

$$C_i = \sum_{r=1}^{s} m^{\mathscr{S}}(X_r)V_{ir} \qquad (15.22)$$

Fig. 15.7 Steps of FCOWA-ER

15.3.3.2 ...to the COWA-ER and FCOWA-ER Approaches

For each alternative A_i, COWA-ER method exploits only the results of the two extreme attitudes (pessimistic and optimistic OWA) jointly [11]. Decision-making under uncertainty is thus based on the M imprecise valuations (or intervals) of expected payoffs gathered in the M-vector $\mathbf{E[C]}$ given by Eq. (15.23).

FCOWA-ER method [12] has been then developed to go beyond two COWA-ER limitations. (1) The BBAs obtained by using α-cuts are consonant support (nested in order) without any correlation between information sources. (2) The computational time for making the combination does not depend on the number M of alternatives.

15.3.3.3 FCOWA-ER-Step 1: Construction of BBAs

Each column in $\mathbf{E[C]}$ is, respectively, normalized to obtain the column-wise normalized expected payoff $\mathbf{E^{Fuzzy}[C]}$ given by Eq. (15.24).

$$\mathbf{E[C]} \triangleq \begin{bmatrix} [C_1^{\min}, C_1^{\max}] \\ [C_2^{\min}, C_2^{\max}] \\ \vdots \\ [C_M^{\min}, C_M^{\max}] \end{bmatrix} \tag{15.23}$$

$$\mathbf{E^{Fuzzy}[C]} \triangleq \begin{bmatrix} N_1^{\min}, N_1^{\max} \\ N_2^{\min}, N_2^{\max} \\ \vdots \\ N_M^{\min}, N_M^{\max} \end{bmatrix} \tag{15.24}$$

The vectors $\mu_1 = [N_1^{\min}, \ldots, N_M^{\min}]$ and $\mu_2 = [N_1^{\max}, \ldots, N_M^{\max}]$ can be seen as two fuzzy membership functions (FMFs) $\mu : A_i \in \mathscr{A} \to [0, 1]$. Given the FoD $\mathscr{A} = \{A_1, A_2, \ldots, A_M\}$, they are, respectively, converted into two BBAs $m_{\text{Pess}}^{\mathscr{A}}(\cdot)$ and $m_{\text{Opti}}^{\mathscr{A}}(\cdot)$ using the α-cut approach [50], considering M as the number of α-cuts.

15.3.3.4 FCOWA-ER-Steps 2 and 3: Combination of the Two BBAs and Decision

The two BBAs $m_{\text{Pess}}^{\mathscr{A}}(\cdot)$ and $m_{\text{Opti}}^{\mathscr{A}}(\cdot)$ are combined with the PCR6 fusion rule. The decision is about choosing $A_{i^\star}$ according to the DM attitude or the minimal distance.

15.4 Application to Efficiency of Torrential Protective Actions

The example previously introduced in Fig. 15.2 is used to show how new belief function-based MCDAs can help DMs to decide on real DMPs. The DMP is first formalized. ER-MCDA, BF-TOPSIS, and FCOWA-ER are then successively applied without detailing computation steps: we only provide inputs and main results.

15.4.1 Formalization of the Decision-Making Problem

The problem is about comparing under uncertainty several torrential protective actions based on their efficiency by involving several criteria.

15.4.1.1 Multi-criteria Decision-Making Problem

Alternatives and decision criteria, with their scoring scales and importance weights, must be specified to provide the structure of the score matrix $\mathbf{S}$ defined by Eq. (15.1).

The set $\mathscr{A}$ gathers $M = 5$ protective alternatives A_i (Fig. 15.2):

- A_1: doing nothing;
- A_2: building check dam series in headwaters;
- A_3: building a sediment trap on the alluvial fan apex;
- $A_4 = A_2 \cup A_3$: building both check dam series and a sediment trap;
- A_5: building individual protections for each element at-risk.

On the one hand, these actions aim at reducing potential damage on elements-at-risk. Several types of damage can occur such as housing destruction or environmental damage due to destruction of dangerous sites, etc. Their assessment in monetary value can strongly be debated as, for example, for human casualties [6]. On the other hand, each alternative involves high investment and maintenance cost.

A DM thus considers $N = 5$ criteria g_j with specific scoring scale X_j [6] to compare alternatives according to their efficiency. He wants to minimize g_1 and g_2 (decreasing preference) and to maximize g_3, g_4, and g_5 (increasing preference), with:

- g_1: investment cost in € ($x_{i1} \in X_1 = \mathbb{R}^+$);
- g_2: annual maintenance cost in € ($x_{i2} \in X_2 = \mathbb{R}^+$);
- g_3: annual risk reduction of damaged houses surface in m^2 ($x_{i3} \in X_3 = \mathbb{R}^+$);
- g_4: annual risk reduction in *human casualties* ($x_{i4} \in X_4 = \mathbb{R}^+$);
- g_5: annual risk reduction in *number* of dangerous sites ($x_{i5} \in X_5 = \mathbb{R}^+$).

In practice, for each A_i and g_j, annual risk reduction $\Delta R_j(A_i)$ is computed as reduction of potential damage expected value: $\Delta R_j(A_i) = R_j(0) - R_j(A_i)$, where $R_j(0)$ is the baseline risk (without A_i) and $R_j(A_i)$ is the residual risk with A_i [6].

Given this set of criteria, DM uses the AHP process [3] to define the 5-vector of their importance weights: $\mathbf{w} = [0.08, 0.04, 0.10, 0.46, 0.32]$.

15.4.1.2 Decision-Making Under Uncertainty

Damage assessment depends on torrential hazards: without flood, there is no damage; during a big but rare flood, damage are higher; damage are much higher with a big debris flows. In practice, several scenarios of torrential hazards are thus taken into account to assess annual risk reduction criteria. They must be specified to provide the structure of the payoff matrix $\mathbf{C}$ defined by Eq. (15.2).

A DM considers a set of states of the nature $\mathscr{S}$ with $K = 7$ scenarios as follows:

- liquid floods without bedload transport : S_1 with $Q_l < Q_{l1}$[4]; S_2 with $Q_l \geqslant Q_{l1}$;
- floods with bedload transport : S_3 with $V_s < V_{s1}$[5]; S_4 with $V_s \geqslant V_{s1}$;
- debris flow : S_5 with $V_l < V_{l1}$[6]; S_6 with $V_{l1} \leqslant V_l < V_{l2}$; S_7 with $V_l \geqslant V_{l2}$

15.4.2 ER-MCDA to Assign an Efficiency Label to Each Alternative

In practice, a first DMP is about assigning each alternative to a qualitative efficiency label. Therefore, ER-MCDA methodology can be used.

15.4.2.1 ER-MCDA Inputs

FoD for decision gathers $n = 4$ exhaustive and mutually exclusive efficiency labels, with $\Theta = \{\theta_1 = \text{no}, \theta_2 = \text{low}, \theta_3 = \text{medium}, \theta_4 = \text{high}\}$. The mapping model of each criterion is provided in Fig. 15.8.

Two sources (experts) s_q, $q = 1, 2$, are assumed totally reliable ($\alpha_1 = \alpha_2 = 1$). For each A_i and g_j, each one provides the possibility distributions $\pi_{q,ij}$ in Table 15.1.

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[4] Q_l = liquid flow.

[5] V_s = solid volume.
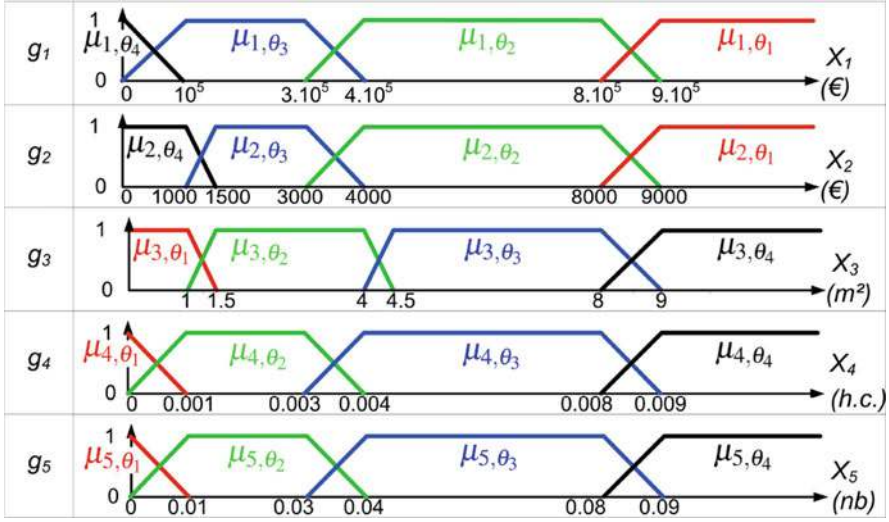
[6] V_l = debris flow volume.

Fig. 15.8 ER-MCDA-input: $N = 5$ mapping models

Table 15.1 ER-MCDA-input: imprecise evaluations $\pi_{q,ij}$ ($q = 1, 2; i = 1, \ldots, 5; j = 1, \ldots, 5$)

A_i	s_q	g_1 [a]	g_2 [a]	g_3 [b]	g_4 [b]	g_5 [b]
A_1	s_1	0,0,0,0	0,0,0,0	0,0	0,0	0,0
A_1	s_2	0,0,0,0	0,0,0,0	0,0	0,0	0,0
A_2	s_1	$(2, 3, 3, 5).10^5$	$(3, 5, 7, 8).10^3$	1.1, 4.3	$(2.1, 8.9).10^{-3}$	$(3.4, 6.4).10^{-2}$
A_2	s_2	$(1, 2, 2, 3).10^5$	$(2, 3, 4, 5).10^3$	2, 5.2	$(1.2, 10.3).10^{-3}$	$(4.3, 8.7).10^{-2}$
A_3	s_1	$(2, 4, 4, 6).10^5$	$(1, 1.5, 2, 3).10^3$	1.2, 4.8	$(2.3, 11.8).10^{-3}$	$(5.1, 8.5).10^{-2}$
A_3	s_2	$(4, 5, 6, 8).10^5$	$(0.5, 1, 1, 2).10^3$	0.1, 5.5	$(3.1, 13.1).10^{-3}$	$(3.7, 8.5).10^{-2}$
A_4	s_1	$(4, 7, 7, 11).10^5$	$(4, 7.5, 9, 11).10^3$	3.3, 8.45	$(3.5, 16.8).10^{-3}$	$(6.2, 9.4).10^{-2}$
A_4	s_2	$(5, 7, 8, 11).10^5$	$(2.5, 4, 5, 7).10^3$	3.1, 8.9	$(3.4, 16.1).10^{-3}$	$(4.3, 9.2).10^{-2}$
A_5	s_1	$(9, 10, 12, 14).10^5$	0, 0, 0, 0	4.2, 9.2	$(3.7, 8.4).10^{-3}$	$(3.1, 8.4).10^{-2}$
A_5	s_2	$(8, 9, 10, 11).10^5$	0, 0, 0, 0	4.65, 8.25	$(2.1, 9.3).10^{-3}$	$(1.5, 9.2).10^{-2}$

[a]Criteria assessed by {a,b,c,d} as shown in Fig. 15.3
[b]Criteria assessed by {a,d} (a = b and c = d)

15.4.2.2 ER-MCDA Results

For each criterion g_j, the new mapping process is used in ER-MCDA-Step 2 to map each $\pi_{q,ij}$ of Table 15.1 into the g_j mapping model of Fig. 15.8. It provides BBAs $m_{1,ij}^{\Theta}(\cdot)$ for s_1 and $m_{2,ij}^{\Theta}(\cdot)$ for s_2. Applying ER-MCDA-Step 3 on those BBAs, $M = 5$ BBAs $m_i^{\Theta}(\cdot)$ are finally obtained in Table 15.2.

To avoid assumption $|\theta_c| = 1$, ER-MCDA-Step 4 is based on computing distances $d_{BI}(m_i^{\Theta}, m_X^{\Theta})$, from each column of Table 15.2. For each A_i, the chosen focal element $\hat{X}(A_i) \in 2^{\Theta}$, the corresponding value of minimal distance $d_{BI}^{\min}$, and the decision quality $q(\hat{X}(A_i))$ are given in Table 15.3.

Table 15.2 ER-MCDA-Step 3 results: BBAs $m_i^{\Theta}(\cdot)$ obtained for each alternative A_i

Focal elements X	$m_1^{\Theta}(\cdot)$	$m_2^{\Theta}(\cdot)$	$m_3^{\Theta}(\cdot)$	$m_4^{\Theta}(\cdot)$	$m_5^{\Theta}(\cdot)$
θ^c	0.00049	0.00075	0.00067	0.00079	0.04062
$\theta_1 \cup \theta^c$	0.96997	0	0	0	0.03157
$\theta_2 \cup \theta^c$	0	0.26680	0.02393	0.02293	0
$\theta_1 \cup \theta_2 \cup \theta^c$	0	0	0	0.00830	0
$\theta_3 \cup \theta^c$	0	0.36178	0.97036	0.71702	0.86412
$\theta_2 \cup \theta_3 \cup \theta^c$	0	0.36113	0.00416	0.02852	0
$\theta_4 \cup \theta^c$	0.02954	0	0.00018	0.17687	0.03597
$\theta_3 \cup \theta_4 \cup \theta^c$	0	0.00954	0.00070	0.04557	0.02772

Table 15.3 ER-MCDA-Step 4 results: decision based on $d_{BI}(m_i^{\Theta}, m_X^{\Theta})$

Alternative A_i	$d_{BI}^{\min}$	$\hat{X}(A_i)$	$q(\hat{X}(A_i))$
A_1	0.0172	$\theta_1 \cup \theta^c$	0.9990
A_2	0.1855	$\theta_2 \cup \theta_3 \cup \theta^c$	0.9878
A_3	0.0154	$\theta_3 \cup \theta^c$	0.9991
A_4	0.1319	$\theta_3 \cup \theta^c$	0.9919
A_5	0.0567	$\theta_3 \cup \theta^c$	0.9968

Solution A_1 of doing nothing is mainly no efficient. Alternative A_2 is lowly or mediumly efficient, while the three other ones are mediumly efficient. It corresponds to a partial preference order: $A_3 \sim A_4 \sim A_5 \succeq A_2 \succ A_1$.

As shown in Table 15.3, quality indicators of decisions are similar for A_3 and A_5 and better than for A_4. For A_2, global mass is more distributed with $m_2^{\Theta}(\theta_3 \cup \theta^c) = 0.36$, $m_2^{\Theta}(\theta_2 \cup \theta_3 \cup \theta^c) = 0.36$, and $m_2^{\Theta}(\theta_2 \cup \theta^c) = 0.27$: decision quality is less good.

15.4.3 BF-TOPSIS to Rank Alternatives

Another DMP is about ranking all potential solutions given previous criteria. BF-TOPSIS methods are compared to help to solve it.

15.4.3.1 BF-TOPSIS Inputs

A precise score of each A_i based on each g_j must be provided to obtain the score matrix $\mathbf{S}$, consistent with possibility distributions $\pi_{q,ij}$ given in Table 15.1. To take into account decreasing preference, initial scores of g_1 and g_2 are multiplied by -1 providing the score matrix $\mathbf{S}^{\mathrm{pref}}$ defined by Eq. (15.25).

Table 15.4 BF-TOPSIS results: relative closeness $C(A_i, A^{\text{best}})$

Alternative A_i	BF-TOPSIS1	BF-TOPSIS2	BF-TOPSIS3	BF-TOPSIS4
A_1	0.12	0.12	0.03	0.03
A_2	0.49	0.49	0.68	0.68
A_3	0.66	0.66	0.85	0.85
A_4	0.69	0.69	0.88	0.88
A_5	0.92	0.92	0.97	0.97

$$
\mathbf{S}^{\text{pref}} \triangleq
\begin{array}{c}
\\ A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5
\end{array}
\begin{pmatrix}
\overset{g_1}{0} & \overset{g_2}{0} & \overset{g_3}{0} & \overset{g_4}{0} & \overset{g_5}{0} \\
-300000 & -6000 & 5 & 0.007 & 0.02 \\
-300000 & -1500 & 5 & 0.008 & 0.04 \\
-600000 & -7500 & 7 & 0.008 & 0.05 \\
-1000000 & 0 & 7 & 0.008 & 0.1
\end{pmatrix}
\tag{15.25}
$$

15.4.3.2 BF-TOPSIS Results

Given $\mathbf{S}^{\text{pref}}$, applying the four BF-TOPSIS methods provides relative closeness $C(A_i, A^{\text{best}})$ of each alternative A_i with the ideal best solution A^{best} in Table 15.4.

In this case, whatever the BF-TOPSIS method used, preference ranking of all alternatives according to descending order of $C(A_i, A^{\text{best}})$ is $A_5 \succ A_4 \succ A_3 \succ A_2 \succ A_1$. In [6], for this application case, authors not only give computation details but also a comparison with classical MCDA methods such as CBA and AHP. They show that CBA is very sensitive to criteria choice for monetary valuation such as the human life (g_4) and that BF-TOPSIS is more robust to rank reversal problems than AHP.

15.4.4 FCOWA-ER to Choose the Best Alternative Under Uncertainty

The final practical DMP is about choosing the best solution to implement, considering the knowledge on the states of the nature.

15.4.4.1 FCOWA-ER Inputs

For each A_i, $i = 1, \ldots, 5$ and each scenario S_k, $k = 1, \ldots, 7$, C_{ik} is the efficiency level in the 5×7 matrix $\mathbf{C}$ defined by Eq. (15.26). It can be extracted after implementing ER-MCDA to solve a previous MCDM problem, given a specific S_k. A quantitative transformation of labels θ_e, $e = 1, \ldots, n$ into [1; 10] (the higher

Table 15.5 FCOWA-ER result: credibility, BetP, $DSmP_{\epsilon=10^{-6}}$, and plausibility of A_i efficiency

A_i	$Bel^{\mathscr{A}}(A_i)$	$BetP^{\mathscr{A}}(A_i)$	$DSmP^{\mathscr{A}}(A_i)_{\epsilon=10^{-6}}$	$Pl^{\mathscr{A}}(A_i)$
A_1	0.000000	0.027908	0.000004	0.139530
A_2	0.000000	0.060282	0.000008	0.269030
A_3	0.000000	0.132010	0.000013	0.470130
A_4	0.000000	0.180390	0.000015	0.566890
A_5	0.404960	0.599420	0.999960	1.000000

is score, the higher is payoff) is proposed. Results of ER-MCDA in Tables 15.2 and 15.3 help give payoffs for S_4.

$$\mathbf{C} = \begin{bmatrix} 9 & 3 & 2 & \mathbf{2} & 1 & 1 & 1 \\ 10 & 8 & 4 & \mathbf{3} & 2 & 1 & 1 \\ 10 & 7 & 6 & \mathbf{6} & 4 & 4 & 1 \\ 10 & 6 & 8 & \mathbf{7} & 3 & 2 & 1 \\ 10 & 8 & 6 & \mathbf{6} & 6 & 5 & 1 \end{bmatrix} \tag{15.26}$$

Expert represents uncertainty on the states of the nature $\mathscr{S}$ through a BBA $m^{\mathscr{S}}(\cdot)$. $s = 4$ focal elements $X_r \in 2^{\mathscr{S}}$ are considered:
$m^{\mathscr{S}}(X_1) = m^{\mathscr{S}}(S_1 \cup S_3 \cup S_5) = 0.4$, $m^{\mathscr{S}}(X_2) = m^{\mathscr{S}}(S_2 \cup S_4 \cup S_5 \cup S_6) = 0.25$,
$m^{\mathscr{S}}(X_3) = m^{\mathscr{S}}(S_7) = 0.1$, $m^{\mathscr{S}}(X_4) = m^{\mathscr{S}}(\mathscr{S}) = 0.25$.
X_1, X_2, and X_3 are partial ignorances, and X_4 is the full ignorance.

15.4.4.2 FCOWA-ER Results

Given payoff matrix $\mathbf{C}$ and BBA $m^{\mathscr{S}}(\cdot)$, applying FCOWA-ER steps, provides the two BBAs $m^{\mathscr{A}}_{\text{Pess}}(\cdot)$ and $m^{\mathscr{A}}_{\text{Opti}}(\cdot)$ for the FoD $\mathscr{A}$ (step 1) which are combined to give the final BBA $m^{\mathscr{A}}_{\text{PCR6}}(\cdot)$ through PCR6 fusion rule (step 2). Table 15.5 shows values of $Bel^{\mathscr{A}}(\cdot)$, $BetP^{\mathscr{A}}(\cdot)$, $DSmP^{\mathscr{A}}_{\epsilon=10^{-6}}(\cdot)$ and $Pl^{\mathscr{A}}(\cdot)$ based on $m^{\mathscr{A}}_{\text{PCR6}}(\cdot)$ (step 3).

Whatever the decision rule, the best action $A_{i\star}$ is always A_5. The total preference ranking is deducted: $A_5 \succ A_4 \succ A_3 \succ A_2 \succ A_1$.

15.5 Conclusions and Perspectives

In practice, torrential risk managers must decide on the best action to reduce damage on elements-at-risk. Therefore, the comparison of efficiency of potential alternatives is generally used. Each one can be assessed through qualitative labels that require a partial ranking of alternatives, but it generally cannot help to choose the best action that requires a total preference ranking.

The decisions are based on several criteria, such as costs and different types of damage reduction, imperfectly assessed under an epistemically uncertain environment corresponding to torrential hazards. Confronted with such difficulties, decisions are generally based on expert knowledge which directly takes into account imperfect information.

This chapter shows how recent developments of MCDAs based on belief function theory actually can help decision-makers in their decision process. A practical example is proposed and new methods, showing their possible combination for a global decision-making was applied.

Whatever the method is used, the first step of any DMP is to define alternatives to compare, criteria to take into account, their importance weights which represent the DM preference between them, and the set of states of the nature. Given the scores of each alternative based on each criterion, the final step is about aggregating this multi-criteria and multi-scenario evaluation to help decision.

The three methods applied in this chapter (ER-MCDA, BF-TOPSIS, FCOWA-ER) are combined at a combination step, while the PCR6 fusion rule is preferred. Nevertheless, analyzing the effect of this choice on the results of each method should be done.

To solve MCDM problems, ER-MCDA helps to take into account imperfect evaluation of criteria potentially provided by several sources. FoD of decision is first specified through qualitative or quantitative labels. Each expert is considered as source who gives a possibility distribution (imprecise scoring) and a mapping model based on fuzzy sets. ER-MCDA makes it possible to choose a label for each alternative providing the quality of this decision.

The four BF-TOPSIS methods help a total preference ranking of all alternatives with a better robustness to rank reversal problems than classical MCDAs. It is based on a precise score matrix representing the MCDM problem. Using intermediary results of ER-MCDA as an intermediary decision step helps to take into account imprecise scoring in BF-TOPSIS.

FCOWA-ER is different from the two previous methods because it proposes a method to solve DMP under uncertainty. It improves the OWA method used when the knowledge of the states of the nature is uncertain. It was first developed to decide given a precise scoring of each payoff. As for BF-TOPSIS, it is possible to apply ER-MCDA for each scenario and to propose a quantitative transformation if qualitative labels are used. It thus helps to take into account initial imperfect scoring.

From an operational point of view, this chapter shows how theoretical methods can help a better formalization of the decision-making process. Indeed, expert and DM elicitation is always needed to express, given the DMP under concern, the criteria to take into account, their importance, and the preferences for their evaluation. Moreover, methods have been applied to DMPs related to protection efficiency, but they are generic and can be applied for any other DMP.

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