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Elodie SUZANNE[1], Nabil ABSI[1], Valeria BORODIN[1], Wilco VAN DEN HEUVEL[2]

[1]Ecole des Mines de Saint-Etienne, CMP Georges Charpak, CNRS UMR 6158 LIMOS, F-13541 Gardanne, France
{elodie.suzanne, absi, valeria.borodin}@emse.fr

[2]Econometric Institute, Erasmus School of Economics, Erasmus University Rotterdam, P.O. Box 1738, 3000 DR Rotterdam, The Netherlands
wvandenheuvel@ese.eur.nl

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The high pace of waste accumulation in landfills and the depletion of scarce natural resources lead us to seek pathways for converting unavoidable production outputs into useful and high added-value products. In this context, we formalize and propose a model for the single-item lot-sizing problem, which integrates the management of unavoidable production residues classified as by-products. During the production process of a main product, a by-product is generated, stored in a limited capacity and transported with a fixed transportation cost. This problem is investigated for two cases of the by-product inventory capacity: time-dependent and constant. We prove the problem with inventory capacities is NP-Hard. To solve it optimally, we develop a pseudo-polynomial time dynamic programming algorithm. For the case with stationary inventory capacities, a polynomial time dynamic programming algorithm is proposed.

Ecole Nationale Supérieure des Mines SAINT-ETIENNE

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Elodie Suzanne[a], Nabil Absi[a], Valeria Borodin[a], Wilco van den Heuvel[b]

[a]Mines Saint-Etienne, Univ Clermont Auvergne, CNRS, UMR 6158 LIMOS, Centre CMP, Departement SFL, F-13541 Gardanne, France
[b]Econometric Institute, Erasmus School of Economics, Erasmus University Rotterdam, P.O. Box 1738, 3000 DR Rotterdam, The Netherlands

Abstract

The high pace of waste accumulation in landfills and the depletion of scarce natural resources lead us to seek pathways for converting unavoidable production outputs into useful and high added-value products. In this context, we formalize and propose a model for the single-item lot-sizing problem, which integrates the management of unavoidable production residues classified as by-products. During the production process of a main product, a by-product is generated, stored in a limited capacity and transported with a fixed transportation cost. This problem is investigated for two cases of the by-product inventory capacity: time-dependent and constant. We prove that the problem with inventory capacities is $\mathcal{NP}$-Hard. To solve it optimally, we develop a pseudo-polynomial time dynamic programming algorithm. For the case with stationary inventory capacities, a polynomial time dynamic programming algorithm is proposed.

Keywords: Circular economy, Production planning, Lot-sizing, By-product, Dynamic programming, Complexity

1. Introduction

Within the umbrella term of *circular economy*, the emergent lines of actions around waste and resource management aim at fostering circular alternatives to predominant linear practices (produce, consume and dispose) in production supply chains. Throughout the world, a growing number of waste prevention and management policies are implemented over the last two decades, in order to promote and support the environment-friendly operations related to the reuse of end-of-life products and recovery of waste materials.

Consistent with these global trends, the *industrial symbiosis* seeks to take sustainable competitive advantage of binding traditionally separate industrial processes in a joint production approach involving physical exchange of production residues or other collateral resources (water, energy, etc.) (Chertow, 2000). In particular, the European Commission promotes the industrial symbiosis as one of the ways to shift toward a circular economy through its programs and action plans[3].

Based on the Waste Framework Directive[1], let us consider a *product* as all lawful material mainly aimed resulting from a production process. This term includes *co-products*, which have the same importance as main products and have their own demand. According to the European Interpretative Communication[2] on waste and by-products, production residues are classified into **(i)** *by-products*, i.e. lawful production residues unavoidably obtained as an integral part of the production process, ready for use without further transformation, whose use is certain and **(ii)** *wastes*, i.e. production residues, which are not by-products. Note that by-products are, by definition, lawful production outputs, whose further use is economically and environmentally sustainable.

The conversion of production residues into by-products, realized by using waste from one industrial process in another one as illustrated in Figure 1, is commonly referred to as *by-product synergy* (Lee, 2012; Lee and Tongarlak, 2017) or *industrial ecosystem* (Herczeg et al., 2018). This paradigm of joint production offers opportunities for all three dimensions of the sustainable development, economic, environmental and social, by: **(i)** avoiding disposal costs and increasing resource efficiency, **(ii)** reducing raw material consumption, **(iii)** supporting the regional economic development. Over the past two decades, an increasing number of industrial symbiosis networks (also called eco-industrial parks) have been implemented all around the world (Herczeg et al., 2018). Let us mention some eloquent specific cases encountered in Europe: **(i)** The Platform for Industry and Innovation at Caban Tonkin[4] (PIICTO) (France) supports the synergies between industrial activities located in the heart of the port of Marseille Fos (exchanges of material and energy flows, pooling of services and equipment), **(ii)** The Kalundborg eco-industrial park[5] (Denmark) regroups separate industries, which use each other's by-products, energy, water and various services, **(iii)** The Deltalinqs[6] (The Netherlands) promotes the industrial ecology to industrial companies located in the Europoort/Botlek harbour area near Rotterdam.

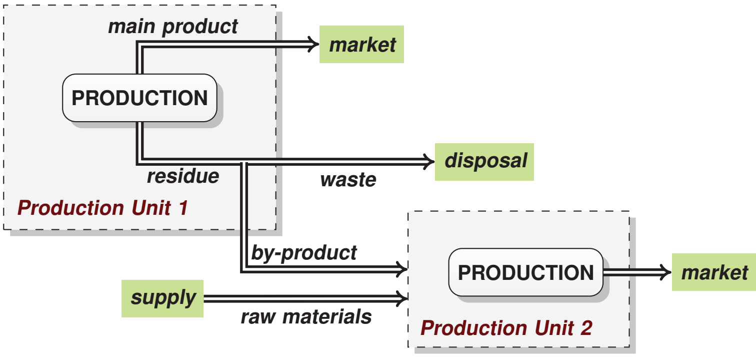

Figure 1: Process flow diagram of by-product synergy

Driven by the nature of the exchanged production streams, setting up industrial symbiosis networks implies multiple requirements including: technological feasibility, organizational and operational coordination, and business framing. In particular, the emergence of these networks raises new production planning problems (Herczeg et al., 2018). To contribute in this direction, the current paper integrates the management of a by-product in the classical lot-sizing problem. More precisely, we introduce and deal with the single-item lot-sizing problem with a by-product storable in limited quantities, by: **(i)** performing a complexity analysis for time-varying and stationary non-decreasing inventory capacities, and **(ii)** proposing structural properties of optimal solutions and exact solution methods based on dynamic programming.

The remainder of this paper is organized as follows. Section 2 reviews the literature related to lot-sizing problems, which integrate the joint production of controlled and uncontrolled outputs. The problem under study is introduced and formalized in Section 3. The lot-sizing problem with a by-product and non-decreasing inventory capacities is examined for two special cases, namely: **(i)** with time-dependent inventory bounds in Section 5, and **(ii)** with constant inventory bounds in Section 6. Complexity analysis, structural properties and solution methods are provided for both of these cases. Finally, this paper ends in Section 7 with conclusions and discussions on future extensions of this work.

2. Related background

This paper focuses on joint production planning of avoidable and unavoidable products and falls within the framework of well-studied economic lot-sizing problems. The reader is referred to the following reviews for learning more about tactical production planning problems under different lens: dynamic lot-sizing problems (Brahimi et al., 2006; Buschkühl et al., 2010; Díaz-Madroñero et al., 2014; Brahimi et al., 2017), capacitated lot-sizing with extensions (Quadt and Kuhn, 2008), lot-sizing integrated with other decision problems (Melega et al., 2018), etc.

In what follows, let us focus on mid-term production planning problems, which take explicitly into account the by-products generated during the production process of primary products. It is worthwhile to mention that, both by-product synergy systems and co-production systems fall within the framework of joint production systems. Despite this, they are different in their scope.

Co-production systems involve the simultaneous production of different co-products in a single run, in the framework of which (Bansal and Transchel, 2014; Bitran and Gilbert, 1994; Santos and Almada-Lobo, 2012):

- Each co-product has its own demand coming from traditional markets;
- The demand of each product can initiate the production process.

By contrast, the core specificity in by-product synergy settings resides in (Lee, 2012, 2016; Lee and Tongarlak, 2017):

- The unavoidable and uncontrollable generation of production residues as an integral part of the production process;
- The status of by-products, which stipulates that a by-product may have an opportunistic demand, but cannot trigger the production process by definition.

In general, the joint production of multiple outputs (including both co-products and production residues) in a single run depends on the physical properties of the process technology, being a specific attribute in a wide range of process industries (Chen et al., 2013). Accordingly, the production capacity/quantity is a crucial parameter/decision for joint production systems, since the quantities of produced outputs are interdependent (Lee, 2016). As different industrial applications found in the literature witness, joint production systems in their broad sense are of significant interest:

- *purely co-production systems:* semiconductor fabrication (Bitran and Gilbert, 1994), pulp and paper industry (Santos and Almada-Lobo, 2012);
- *production systems including the management of by-products:* semiconductor fabrication (Rowshannahad et al., 2018), metal processing (Spengler et al., 1997), glass manufacturing (Taşkın and Ünal, 2009).

In the related literature, co-production planning problems have been treated as extensions of lot-sizing problems with demands of co-products different in kind: substitutable (Bitran and Leong, 1995), and not substitutable (Bitran and Gilbert, 1994; Bitran and Leong, 1992; Ağralı, 2012; Lu and Qi, 2011). Both exact solution methods (Ağralı, 2012) and heuristic algorithms (Lu and Qi, 2011; Bitran and Gilbert, 1994; Bitran and Leong, 1992, 1995) are available for these problems.

As previously highlighted, production planning problems with by-products cannot be reduced or generalized to co-production planning problems. To the best of our knowledge, only Sridhar et al. (2014) studied a generic non-linear production problem with by-products, by assuming that the ratio of undesirable by-products increases monotonically as a nonconvex function of the cumulative production mixture. Given the actual circular economy concerns and real-life needs laid out by industrial applications (see e.g. Spengler et al. (1997); Taşkın and Ünal (2009)), this paper intends to strengthen the academic efforts on this topic, by investigating a generic single-item lot-sizing problem with a by-product storable in limited quantities.

3. Mathematical formulation

Consider a single-item lot-sizing problem dealing with a by-product and inventory capacities on the by-product, called ULS-B for short in the rest of this paper. Let us define the ULS-B problem as illustrated in Figure 2, i.e.: Over a planning horizon of T periods, determine when and how much to produce X_t units of a main product at a low cost, while satisfying a deterministic demand d_t and forwarding the amount of the by-product generated with a proportion of $\alpha \in \mathbb{R}^+$ to a further destination, $\forall t \in \{1, 2, \ldots, T\}$.

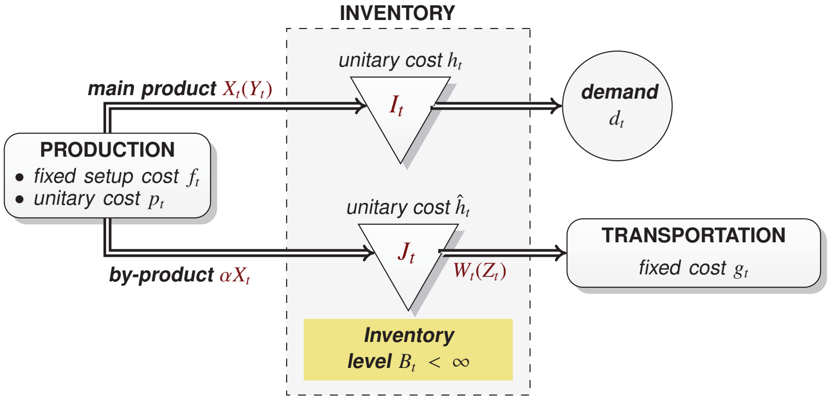

Figure 2: ULS-B problem: *Flow of products*, $\forall t \in \{1, 2, \ldots, T\}$

The production system involves a fixed setup cost f_t and a unitary production cost p_t per period of time. The surplus amount of the produced main product can be kept in inventory at a cost h_t from period t to period $t+1$. To stock the unavoidable by-product, a cost $\hat{h}_t$ is charged per unit and period of time. For the end of each period t, the inventory level of the by-product is limited to B_t. The by-product transportation is performed at a fixed cost g_t. Note that a transportation operation implies the complete emptying of the by-product inventory. This remark is always valid because of positive costs.

Before proceeding to the problem modeling, let us consider the following assumptions:

- As commonly assumed in the related literature, the bounded inventory capacities B_t are non-decreasing during the planning horizon, $t \in \{1, 2, ..., T\}$. If this is not the case, the respective capacities can be pretreated and reduced to the case of a non-decreasing capacity series (see e.g. Atamtürk and Küçükyavuz (2008)). Note that, this assumption is realistic in many industrial settings. There is no reason for a company, which has invested in a warehouse or a reservoir, not to use it later.

- Both inventory levels of the main product and the by-product are assumed null at the end of the planning horizon.

Three main decisions are posed by the ULS-B problem: **(i)** when and **(ii)** how much to produce, as well as **(iii)** when to transport. Accordingly, all other related decisions are implied, namely: inventory levels of the main product and the by-product, as well as the quantities to transport.

Hence, let X_t and W_t be two decision variables that represent the amount of the main product to produce at period t, and respectively, the amount of the by-product to transport at the end of period t. Denote by Y_t and Z_t two binary decision variables that indicate if the production of the main product takes place in period t, and respectively, if the transportation of the by-product is performed at the end of period t. The inventory levels of the main product and the by-product at the end of period t are represented by I_t and J_t, $t \in \{1, 2, ..., T\}$.

Table 1: ULS-B problem: *Notations*

Parameters:	
T	number of time periods indexed by $t \in \{1, 2, \ldots, T\}$
α	production coefficient of the by-product (w.l.o.g., $\alpha := 1$)
d_t	demand of the main product in period t
B_t	inventory capacity of the by-product in period t
p_t	production cost of the main product in period t
f_t	fixed setup cost of the main product in period t
h_t	holding cost of the main product at the end of period t
$\hat{h}_t$	holding cost of the by-product at the end of period t
g_t	fixed transportation cost for the by-product at the end of period t
M_t	big number with $M_t = \min\left\{\sum_{i=t}^{T} d_i; \frac{B_t}{\alpha}\right\}$
Decision variables:	
X_t	production quantity of the main product in period t
Y_t	binary setup indicator for production of the main product in period t
I_t	inventory level of the main product at the end of period t
J_t	inventory level of the by-product at the end of period t
W_t	transportation quantity of by-product at the end of period t
Z_t	binary setup indicator for transportation of the by-product at the end of period t

By making use of the notation summarized in Table 1, the ULS-B problem can be formulated as a mixed integer linear program as follows:

$$\min \sum_{t=1}^{T} \left(p_t X_t + f_t Y_t + h_t I_t + \hat{h}_t J_t + g_t Z_t\right) \tag{1}$$

$$\text{s.t. } I_{t-1} + X_t - I_t = d_t, \qquad \forall t \in \{1, 2, \ldots, T\} \tag{2}$$

$$J_{t-1} + \alpha X_t - J_t = W_t, \qquad \forall t \in \{1, 2, \ldots, T\} \tag{3}$$

$$W_t \leq B_t Z_t, \qquad \forall t \in \{1, 2, \ldots, T\} \tag{4}$$

$$J_t \leq B_t(1 - Z_t), \qquad \forall t \in \{1, 2, \ldots, T\} \tag{5}$$

$$X_t \leq M_t Y_t, \qquad \forall t \in \{1, 2, \ldots, T\} \tag{6}$$

$$I_0 = J_0 = I_T = J_T = 0, \tag{7}$$

$$Y_t, Z_t \in \{0, 1\}, \qquad \forall t \in \{1, 2, \ldots, T\} \tag{8}$$

$$X_t, I_t, J_t, W_t \geq 0, \qquad \forall t \in \{1, 2, \ldots, T\} \tag{9}$$

The objective function (1) minimizes the sum of the following costs: production (fixed and variable), inventory holding and transportation. The set of equalities (2) models the flow conservation constraints of the main product. By the same token, constraints (3) express the flow conservation of the by-product. Linking constraints (4) involve a fixed transportation cost, if the accumulated by-product is transported in that period. Constraints (5) empty the by-product inventory level when a transport operation is triggered. In case of production, inequalities (6) ensure that a fixed setup cost is paid. Without loss of generality, constraints (7) set initial and ending inventories to zero. Non-negativity and binary requirements are given via constraints (8)-(9).

Remark 1. *Given constraints* (3)-(5)*, X_t cannot be greater than $\frac{B_t}{\alpha}$. Without loss of generality, let us suppose in what follows that $\alpha = 1$, by setting $B_t := \frac{B_t}{\alpha}, \forall t \in \{1, 2, \ldots, T\}$.*

4. ULS-B problem: Network flow representation

A solution of a ULS-B problem can be represented by a flow in a special network, as shown in Figure 3. In such a network, black nodes depict time periods. The lower part of the network, depicted with solid lines, represents the flow balance equations of the main product. Solid horizontal arcs represent the inventory of the main product. Demands are expressed by the solid inclined outgoing arcs. The upper part of the network, in dashed lines, refers to the flow balance equations of the by-product. Dashed horizontal arcs represent the inventory of the by-product. Dashed outgoing arcs represent the transportation periods with the associated by-product inventory levels. The absence of horizontal arcs means null inventory levels. The particularity of this network lies in the fact that the flow generated in a production period t is the same in both directions (upper and lower arcs) and is equal to the quantity produced in period $t, t \in \{1, 2, \ldots, T\}$.

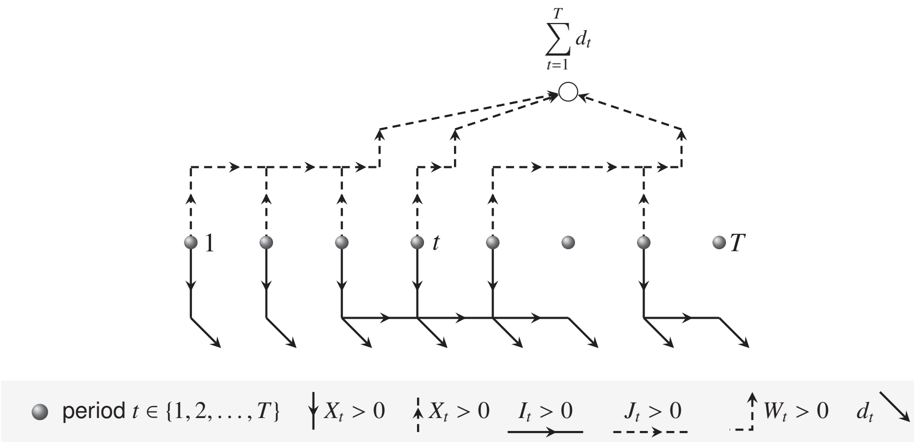

Figure 3: ULS-B problem: *Network flow representation*

Property 1. *There exists an optimal solution of the ULS-B problem, such that arcs corresponding to variables with flows strictly between their lower and upper bounds form a cycle free network.*

Proof. In order to proof this property, we modify the network flow defined previously (see Figure 3). To do so, let us reverse the arcs of the by-product flows (dashed lines). Contrary to the original network flow model in which production flows are doubled, the new obtained network is a classical one. The source node is the one on the top, it injects $\sum_{t=1}^{T} d_t$ into the network flow. The flow generated by this quantity traverses the arcs of this new network in order to satisfy demands of all periods.

In classical network flow models with concave costs, extreme flows are cycle free (see for example Ahuja et al. (1988); Zangwill (1968)). Since the ULS-B problem can be modeled as a classical network flow problem with concave costs, extreme solutions are cycle free. Recall that extreme flows are those that cannot be represented as a convex combination of two distinct flows. We also know that there exists at least an optimal solution of the ULS-B among these extreme flows. This concludes the proof. □

Note that in the ULS-B, an extreme solution $S = (X, I, J, W)$ can contain at most two cycle configurations, which differ one from another by the nature of by-product flows, other than those related to production. Accordingly, let us distinguish the following two cases illustrated in Figure 4:

Case.1 Solution S contains a cycle including inventory by-product flows defined between periods k and ℓ with $k < \ell, k, \ell \in \{1, 2, \cdots, T\}$.

Case.2 Solution S contains a cycle including transportation by-product flows defined between periods i and j with $i < j, i, j \in \{1, 2, \cdots, T\}$.

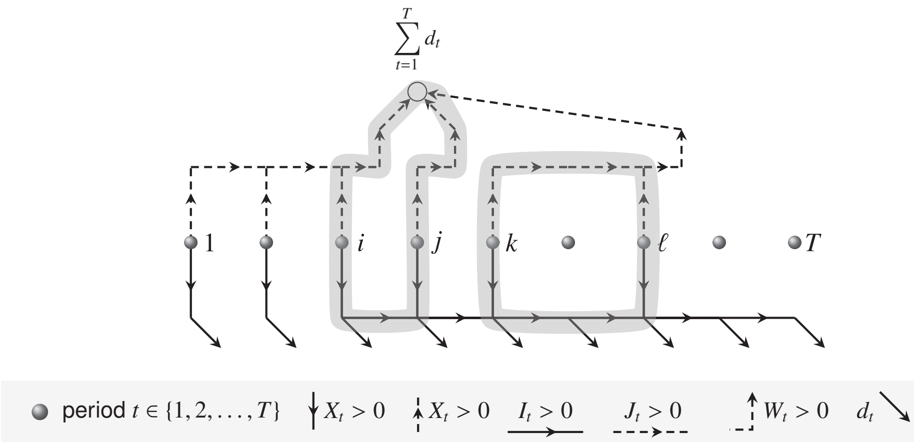

Figure 4: Cycle configurations in the ULS-B network flow

5. ULS-B problem: The general case

In what follows, let us investigate the ULS-B problem formalized in the previous section in the general case. Recall that the inventory capacities of the by-product are non-decreasing.

5.1. Complexity of the ULS-B problem

Theorem 1. *The ULS-B problem with time-dependent inventory capacities of the by-product is $\mathcal{NP}$-Hard.*

Proof. The proof of $\mathcal{NP}$-Hardness of ULS-B is performed by reduction from the capacitated lot-sizing (CLS) problem, the general case of which is known to be $\mathcal{NP}$-Hard (Florian et al., 1980). The decision version of the CLS problem is defined by:

- a planning horizon of $\{1, 2, \ldots, \tilde{T}\}$,
- limited production capacities $\tilde{C}_t$, $\forall t \in \{1, 2, \ldots, \tilde{T}\}$,
- demands $\tilde{d}_t$, $\forall t \in \{1, 2, \ldots, \tilde{T}\}$,
- a triple cost: fixed setup $\tilde{f}_t$, unit production cost $\tilde{p}_t$, and unit inventory holding cost $\tilde{h}_t$, $\forall t \in \{1, 2, \ldots, \tilde{T}\}$.

Let $\tilde{X} = (\tilde{X}_1, \tilde{X}_2, \ldots, \tilde{X}_T)$ be the vector of produced quantities, and $\tilde{I} = (\tilde{I}_1, \tilde{I}_2, \ldots, \tilde{I}_T)$ be the vector of inventory levels during the planning horizon. Denote by $\tilde{Y} = (\tilde{Y}_1, \tilde{Y}_2, \ldots, \tilde{Y}_T)$ the production indicator vector. The question posed by the capacitated lot-sizing problem is: Does there exist a production plan $(\tilde{X}, \tilde{I}, \tilde{Y})$ of total cost at most equal to a given value V, which satisfies demands $\tilde{d} = (\tilde{d}_1, \tilde{d}_2, \ldots, \tilde{d}_T)$?

An instance I^{CLS} of the capacitated lot-sizing problem can be transformed into an instance I of ULS-B by making the following substitutions $\forall t \in \{1, 2, \ldots, \tilde{T}\}$:

(S.1) Number of periods: $T = \tilde{T}$;

(S.2) Demands: $d_t = \tilde{d}_t$;

(S.3) Capacities: $B_t = \tilde{C}_t$;

(S.4) Costs related to the main product: $f_t = \tilde{f}_t$, $p_t = \tilde{p}_t$ and $h_t = \tilde{h}_t$;

(S.5) Costs related to the by-product: $g_t = 0$ and $\hat{h}_t = 0$.

Let us show that instance I^{CLS} has an affirmative answer, if and only if, there exists a feasible solution (X, Y, I, J, W, Z) for instance I such that $\sum_{t=1}^{T} \left(p_t X_t + f_t Y_t + h_t I_t + \hat{h}_t J_t + g_t Z_t\right) \leq V$. To do this, we prove the conditional relationship between CLS and ULS-B problems related to the solution existence.

($\Longrightarrow$). Suppose that instance I^{CLS} has an affirmative answer. Let $(\tilde{X}, \tilde{I}, \tilde{Y})$ be a production plan, such that $\sum_{t=1}^{\tilde{T}} \left(\tilde{p}_t \tilde{X}_t + \tilde{f}_t \tilde{Y}_t + \tilde{h}_t \tilde{I}_t\right) \leq V$. A feasible solution (X, Y, I, J, W, Z) for instance I, such that the total cost is at most equal to V, can be built as follows: **(i)** produce $X = \tilde{X}$ quantities of the main product, this generates by-products quantities less than $B = (B_1, B_2, \ldots, B_T)$ by virtue of substitution (S.3), **(ii)** hold $I = \tilde{I}$ levels of the main product, and **(iii)** transport at the end of each period the entire generated quantity of the by-product during this period. Given substitutions (S.1)-(S.5), it follows that $\sum_{t=1}^{T} \left(p_t X_t + f_t Y_t + h_t I_t + \hat{h}_t J_t + g_t Z_t\right) \leq V$.

($\Longleftarrow$). Conversely, assume that instance I has a positive answer, i.e. there exists a production plan (X, Y, I, J, W, Z), which satisfies all demands with a cost at most equal to V. Making use of substitutions (S.1)-(S.5), it can immediately be checked that $\sum_{t=1}^{\tilde{T}} \left(\tilde{p}_t \tilde{X}_t + \tilde{f}_t \tilde{Y}_t + \tilde{h}_t \tilde{I}_t\right) \leq V$, where $\tilde{X} = X$, $\tilde{I} = I$ and $\tilde{Y} = Y$. □

5.2. Definitions and structural properties of optimal solutions

In this section, let us give several definitions and derive a useful structural property of optimal solutions.

Definition 1 (Production period)**.** *A period $t \in \{1, 2, \ldots, T\}$ is called **a production period**, if the production of the main product is performed at this period, i.e. $X_t > 0$.*

Definition 2 (Inventory period)**.** *A period $t \in \{1, 2, \ldots, T\}$ is called **an inventory period**, if at least one of the following conditions holds:*

- *The inventory level of the main product at the end of period t is equal to zero, i.e. $I_t = 0$.*

- *The inventory level of the by-product at the end of period t is equal to zero, i.e.* $J_t = 0$.
- *The inventory level of the by-product at the end of period t is equal to its maximum inventory capacity, i.e.* $J_t = B_t$.

Definition 3 (Transportation period)**.** *A period* $t \in \{1, 2, \dots, T\}$ *is called* ***a transportation period****, if the transportation of the by-product takes place at the end of this period, i.e.* $W_t > 0$.

Note that, ULS-B is not a classical bi-level lot-sizing problem. The production of the main product causes the uncontrolled generation of a by-product. Hence, a production period refers not only to the production of the main product, but also reveals the generation of a by-product. Based on the above definitions and consistent with the network structure of ULS-B, the following property generalizes the classical block decomposition approach introduced by Florian and Klein (1971).

Property 2. *In an optimal solution of the ULS-B problem, there is at most one production period between two consecutive inventory periods.*

Proof. Consider an optimal solution of ULS-B containing two consecutive inventory periods $j-1, \ell \in \{2, 3, \dots, T\}$. By virtue of Definitions 1-2, period j is either **(i)** a production period with $X_j > 0$ and $j \neq \ell$, or **(ii)** an inventory period with $X_j = 0$ and $j = \ell$, since it inherits one of the inventory period conditions from its predecessor.

Let $j \neq \ell$ be a production period. Suppose, by contradiction, that there exists another production period t between j and ℓ. As $j-1$ and ℓ are two consecutive inventory periods, the inventory levels of both the main product and the by-product are not null and do not reach the inventory capacity until t. The flows corresponding to the production in periods j and t form thus a cycle. From Property 1, it follows that the considered solution is not optimal. Hence, the assumption that there exists more than one production period between two consecutive inventory periods is false. □

5.3. Dynamic programming algorithm

This section presents a pseudo-polynomial dynamic programming algorithm for solving the general case of ULS-B. Property 2 is used to reduce the number of generated states. For the sake of convenience, let us introduce the notion of block to characterize the extreme solutions of the problem under study.

Definition 4 (Block)**.** *The set of time periods* $\{j, j+1, \dots, \ell\}$ *between two consecutive inventory periods* $j-1$ *and* ℓ *is called a* ***block****, where* $j \leq \ell$, $\forall j, \ell \in \{1, 2, \dots, T\}$.

In other words, a block formed by two inventory periods j and ℓ implies that: **(i)** $I_{j-1} = 0$ or $J_{j-1} \in \{0, B_{j-1}\}$, **(ii)** $I_\ell = 0$ or $J_\ell \in \{0, B_\ell\}$, and **(iii)** $I_t > 0$ and $0 < J_t < B_t$, $\forall t \in \{j, \dots, \ell-1\}$. As shown in Property 2, there is at most one production period k between two consecutive inventory periods $j-1$ and ℓ, $j-1 < k \leq \ell$. Hence, let us represent an extreme solution as a sequence of blocks:

$$[j, k, \ell]^{\omega\gamma}_{\beta\delta},$$

where values ω, γ, β and δ indicate the states of entering and ending inventory levels of both the main product and by-product.

More specifically, a block $[j, k, \ell]^{\omega\gamma}_{\beta\delta}$ is defined by two consecutive inventory periods $j-1$ and ℓ, and at most one production period k is characterized by the following states:

- $\omega \in \{0, 1, \cdots, \tilde{M}_j\}$: entering inventory level of the main product at period j, with $\tilde{M}_t = \sum_{i=t}^{T} d_i$, $\forall t \in \{1, \cdots, T\}$ and $\tilde{M}_{T+1} = 0$;

- $\gamma \in \{0, 1, \cdots, \tilde{M}_{\ell+1}\}$: ending inventory level of the main product at period ℓ;
- $\beta \in \{0, 1, \cdots, N_j\}$: entering inventory level of the by-product of period j, with $N_j = B_{j-1}, \ \forall j \in \{1, \cdots, T\}$ and $N_{T+1} = 0$.
- $\delta \in \{0, 1, \cdots, R_\ell\}$: ending inventory level of the by-product of period ℓ, with $R_\ell = B_\ell, \ \forall \ell \in \{1, \cdots, T-1\}$ and $R_T = 0$.

By convention, if there is no production period between j and ℓ, k is set to 0. The objective value of the optimal policy for a block $[j, k, \ell]^{\omega\gamma}_{\beta\delta}$ is denoted $\varphi^{\omega\gamma}_{\beta\delta}(j, k, \ell)$.

Define the following three functions in order to improve the readability of the subsequent equations:

- The function $\mathcal{P}_t(Q)$ provides the cost of producing the quantity Q at period t:

$$\mathcal{P}_t(Q) = \begin{cases} f_t + p_t Q, & \text{if } Q > 0 \text{ and } t > 0 \\ 0, & \text{if } Q = 0 \text{ or } t = 0 \\ +\infty, & \text{if } Q < 0. \end{cases}$$

- The function $\mathcal{H}_{t,\ell}(Q)$ calculates the inventory cost of storing the quantity Q of the main product between period t and period ℓ, while considering the demand satisfaction between these two periods:

$$\mathcal{H}_{t,\ell}(Q) = \begin{cases} \displaystyle\sum_{i=t}^{\ell-1} h_i(Q - d_{ti}), & \text{if } Q \geq d_{t,\ell-1} \\ +\infty, & \text{if } Q < d_{t,\ell-1} \end{cases}$$

where $d_{t\ell} = \begin{cases} \sum_{i=t}^{\ell} d_i, & \text{if } t \leq \ell \\ d_{t\ell} = 0, & \text{if } t > \ell. \end{cases}$

- The function $\hat{\mathcal{H}}_{t,\ell}(Q)$ represents the inventory cost of storing the quantity Q of the by-product between period t and period ℓ.

$$\hat{\mathcal{H}}_{t,\ell}(Q) = \begin{cases} \displaystyle\sum_{i=t}^{\ell-1} \hat{h}_i Q, & \text{if } Q \geq 0 \\ +\infty, & \text{if } Q < 0. \end{cases}$$

Given a block $[j, k, \ell]^{\omega\gamma}_{\beta\delta}$, the value of δ can be expressed in terms of ω, γ, β and $d_{j\ell}$: $\delta = \beta + d_{jl} + \gamma - \omega$. Note that if transportation occurs at period l then $\delta = 0$.

By making use of the notations introduced above, the cost of the block $\varphi^{\omega\gamma}_{\beta\delta}(j, k, \ell)$ is given by:

$$\varphi^{\omega\gamma}_{\beta\delta}(j, k, \ell) = \begin{cases} \mathcal{P}_k(d_{j\ell} + \gamma - \omega) + \mathcal{H}_{j,k}(\omega) + \mathcal{H}_{k,\ell+1}(d_{k\ell} + \gamma) + \\ \qquad + \hat{\mathcal{H}}_{j,k}(\beta) + \hat{\mathcal{H}}_{k,\ell+1}(\delta), & \text{if } d_{j,k-1} < \omega \text{ and } \delta \leq B_k \\ +\infty, & \text{otherwise.} \end{cases}$$

According to Property 2, transportation can only occur at the end of period ℓ. In the contrary case, if transportation occurs at a period $m < \ell$, an inventory period is thus created at period m, fact that breaks down the block structure of $[j, k, \ell]^{\omega\gamma}_{\beta\delta}$. Then,

$$\varphi^{\omega\gamma}_{\beta 0}(j, k, \ell) = \min_{0 < \delta \leq R_k} \varphi^{\omega\gamma}_{\beta\delta}(j, k, \ell) + g_\ell - \hat{\mathcal{H}}_{\ell,\ell+1}(\delta)$$

Let $v_t^{\omega\beta}$ be the optimal value of the problem over the periods $[t, T]$, given that $I_{t-1} = \omega$ and $J_{t-1} = \beta$.

Remark 2. *Given constraint (7) of the ULS-B problem, let us initialize the initial and final states of main and by-product inventory level as follows:*

- *Initial states: $I_0 = J_0 = 0$, by setting $B_0 = 0$;*
- *Final states: $I_T = J_T = 0$, by setting $R_T = 0$ and $\tilde{M}_{T+1} = N_{T+1} = 0$.*

Proposition 1. *For initial null inventory levels of the main product and the by-product, the optimal cost of the ULS-B problem is equal to v_1^{00} given by the last step of the following algorithm, which proceeds backward in time from period T to 1, $\forall j \in \{T, T-1, \cdots, 1\}$:*

$$v_{T+1}^{\omega\beta} = 0, \forall \omega \in \{0, 1, \ldots, \tilde{M}_{T+1}\}, \forall \beta \in \{0, 1, \ldots, N_{T+1}\}$$

$$\begin{aligned}
v_j^{\omega 0} &= \min_{j \le \ell \le T} \min_{j \le k \le \ell} \left\{ \min_{0 \le \gamma \le \tilde{M}_{\ell+1}} \left\{ \varphi_{00}^{\omega\gamma}(j,k,\ell) + v_{\ell+1}^{\gamma 0}, \quad \varphi_{0B_\ell}^{\omega\gamma}(j,k,\ell) + v_{\ell+1}^{\gamma B_\ell} \right\}, \min_{0 \le \delta \le R_k} \left\{ \varphi_{0\delta}^{\omega 0}(j,k,\ell) + v_{\ell+1}^{0\delta} \right\} \right\} \\
v_j^{\omega B_{j-1}} &= \min_{j \le \ell \le T} \min_{j \le k \le \ell} \left\{ \min_{0 \le \gamma \le \tilde{M}_{\ell+1}} \left\{ \varphi_{B_{j-1}0}^{\omega\gamma}(j,k,\ell) + v_{\ell+1}^{\gamma 0}, \quad \varphi_{B_{j-1}B_\ell}^{\omega\gamma}(j,k,\ell) + v_{\ell+1}^{\gamma B_\ell} \right\}, \min_{0 \le \delta \le R_k} \left\{ \varphi_{B_{j-1}\delta}^{\omega 0}(j,k,\ell) + v_{\ell+1}^{0\delta} \right\} \right\}. \\
v_j^{0\beta} &= \min_{j \le \ell \le T} \min_{j \le k \le \ell} \left\{ \min_{0 \le \gamma \le \tilde{M}_{\ell+1}} \left\{ \varphi_{\beta 0}^{0\gamma}(j,k,\ell) + v_{\ell+1}^{\gamma 0}, \quad \varphi_{\beta B_\ell}^{0\gamma}(j,k,\ell) + v_{\ell+1}^{\gamma B_\ell} \right\}, \min_{0 \le \delta \le R_k} \left\{ \varphi_{\beta\delta}^{00}(j,k,\ell) + v_{\ell+1}^{0\delta} \right\} \right\}
\end{aligned} \tag{10}$$

where $0 \le \omega \le \tilde{M}_j, 0 \le \beta \le N_j$ with $B_0 = 0, d_{T+1} = 0$ and $B_{T+1} = 0$.

Proof. By virtue of Property 3, an optimal solution of the ULS-B problem can be partitioned in a series of consecutive blocks. Hence, from the perspective of a given period j, the cost of ULS-B over the periods $[j, T]$ can be recursively expressed via the sum of: **(i)** the cost of the block, starting at period j and finishing at a period ℓ, and **(ii)** the cost of ULS-B over the periods $[\ell+1, T]$, where $j \le \ell \le T$.

A block is defined by two consecutive inventory periods, each having to respect one of the three conditions specified in Definition 2. Hence, at each period $j \in \{T, T-1, \ldots, 1\}$:

- The recursion formula (10) considers all possible couples of inventory level states:
  - $(\omega, 0)$: when transportation is performed at the end of period $j-1$, i.e. $I_{j-1} = \omega$ and $J_{j-1} = 0$;
  - $(0, \beta)$ and (ω, B_{j-1}): when transportation is not performed at the end of period $j-1$, i.e. $I_{j-1} = 0$ and $J_{j-1} = \beta$, or $I_{j-1} = \omega$ and $J_{j-1} = B_{j-1}$, $0 \le \omega \le \tilde{M}_j, 0 \le \beta \le N_j$.

  and, respectively, the corresponding optimal costs are evaluated in recursion formula (10): $v_j^{\omega 0}$, $v_j^{\omega B_{j-1}}$ and $v_j^{0\beta}$.
- For each couple of inventory level states $(\bullet, \bullet)$, all of three possible subsequent block structures are examined by the recursion formula (10): $[j,k,\ell]_{\bullet 0}^{\bullet\gamma}$, $[j,k,\ell]_{\bullet B_\ell}^{\bullet\gamma}$, and $[j,k,\ell]_{\bullet\delta}^{\bullet 0}$.

Given the exhaustive evaluation of all blocks performed via the recursion formula (10), it follows by induction that:

$$v_1^{00} = \min_{1 \le \ell \le T} \min_{1 \le k \le \ell} \left\{ \min_{0 \le \gamma \le \tilde{M}_{\ell+1}} \left\{ \varphi_{00}^{0\gamma}(1,k,\ell) + v_{\ell+1}^{\gamma 0}, \varphi_{0B_\ell}^{0\gamma}(1,k,\ell) + v_{\ell+1}^{\gamma B_\ell} \right\}, \min_{0 \le \delta \le R_k} \left\{ \varphi_{0\delta}^{00}(1,k,\ell) + v_{\ell+1}^{0\delta} \right\} \right\}$$

provides the optimal cost of the ULS-B problem. □

Proposition 2. *The optimal value v_1^{00} can be computed in $O(T^3(\bar{B}+T\bar{d})^2)$, given the pre-calculation of all possible blocks, which can be done in $O\Big(T^3(T^2\bar{d}^2 + T\bar{d}\bar{B} + \bar{B}^2)\Big)$, where $\bar{d} = \dfrac{\sum_{i=1}^{T} d_i}{T}$ and $\bar{B} = \dfrac{\sum_{i=1}^{T} B_i}{T}$.*

Proof. The cost of each block $\varphi_{\beta\delta}^{\omega\gamma}(j,k,\ell)$ presupposes the evaluation of three functions $\mathcal{P}(\cdot)$, $\mathcal{H}(\cdot)$ and $\hat{\mathcal{H}}(\cdot)$, which can be evaluated in linear time $O(T)$ for any fixed parameters $\omega \in [0, \tilde{M}_j], \gamma \in [0, \tilde{M}_{\ell+1}], \beta \in [0, N_j], \delta \in [0, R_l]$ and each fixed triplet (j,k,ℓ):

- Being an affine function, $\mathcal{P}(\cdot)$ is calculated in constant time.
- Functions $\mathcal{H}(\cdot)$ and $\hat{\mathcal{H}}(\cdot)$ can be evaluated in constant time with pre-calculated terms in $O(T^2)$.

Maximum values of ω and γ are lower than $T\bar{d}$, while values of β and δ are lower than $B_t, \forall t \in \{1,2,\ldots,T\}$ and hence on average lower than $\bar{B}$. Since a block is defined by two inventory periods, each having to respect one of the three conditions given in Definition 2, nine blocks structures can be evaluated before the execution of the dynamic programming algorithm for each fixed triplet (j,k,ℓ) in:

- $O(T^2\bar{d}^2)$: $\varphi_{00}^{\omega\gamma}(j,k,\ell)$, $\varphi_{0B_\ell}^{\omega\gamma}(j,k,\ell)$, $\varphi_{B_{j-1}0}^{\omega\gamma}(j,k,\ell)$ and $\varphi_{B_{j-1}B_\ell}^{\omega\gamma}(j,k,\ell)$;
- $O(T\bar{d}\bar{B})$: $\varphi_{0\delta}^{\omega 0}(j,k,\ell)$, $\varphi_{B_{j-1}\delta}^{\omega 0}(j,k,\ell)$, $\varphi_{\beta 0}^{0\gamma}(j,k,\ell)$ and $\varphi_{\beta B_\ell}^{0\gamma}(j,k,\ell)$;
- $O(\bar{B}^2)$: $\varphi_{\beta\delta}^{00}(j,k,\ell)$.

To sum up, the evaluation of all blocks can be preprocessed in $O\Big(T^3(T^2\bar{d}^2 + T\bar{d}\bar{B} + \bar{B}^2)\Big)$, while the overall complexity of computing $v_j^{\omega 0}$, $v_j^{\omega B_{j-1}}$ and $v_j^{0\beta}$, for all $j \in \{T, T-1, \ldots, 1\}, \omega \in [0, \tilde{M}_{j-1}], \beta \in [0, N_j]$ amounts to $O(T^3(\bar{B} + T\bar{d})^2)$. □

Remark 3. *The complexity of the dynamic programming algorithm given in Proposition 1 is pseudo-polynomial. Together with Theorem 1, it proofs that the ULS-B problem is weakly $\mathcal{NP}$-Hard.*

6. ULS-B problem with a stationary capacity

In this section, let us examine a special case of the ULS-B problem with a stationary capacity of the by-product inventory level over the planning horizon, called ULS-B^{const}. This constant capacity is denoted by B in the rest of this section.

6.1. Structural properties of optimal solutions

For convenience, consider the following definitions.

Definition 5 (Fractional transportation period)**.** *A period $t \in \{1,2,\ldots,T\}$ is called **a fractional transportation period**, if $0 < W_t < B$.*

Definition 6 (Full transportation period)**.** *A period $t \in \{1,2,\ldots,T\}$ is called **a full transportation period**, if $W_t = B$.*

Definition 7 (Full inventory period)**.** *A period $t \in \{1,2,\ldots,T\}$ is called **a full inventory period**, if the inventory levels of the main product and the by-product are equal to zero at the end of period t, i.e. $I_t = 0$ and $J_t = 0$.*

Property 3. *In an optimal solution of the ULS-B^{const} problem, there exists at most one fractional transportation period between two consecutive full inventory periods.*

Proof. The proof is done by contradiction. Let $j-1$ and ℓ be two consecutive full inventory periods in an optimal solution $S = (X, I, J, W)$. Suppose by contradiction that between periods $j-1$ and ℓ there exists two fractional transportation periods t and v such that $j-1 < t < v \leq \ell$. This supposition implies that $I_{j-1} = J_{j-1} = 0$, $I_\ell = J_\ell = 0$, $0 < W_t < B$ and $0 < W_v < B$.

Let u be the last production period before transportation period v. Consistent with the connectivity of the network flow corresponding to optimal solution S, there exists a path between:

- periods u and v, which contains flows strictly between their lower and upper bounds, since u and v are not full inventory periods, i.e. $0 < J_i < B, \forall i \in \{u, u+1, \ldots, v-1\}$;
- periods t and u.

The path traversing flows associated with periods t, u and v forms a cycle with the flows emanated from transportation periods t and v, as illustrated in Figure 5. In particular, this path contains flows strictly between their lower and upper bounds, which contradicts Property 1 verified by the optimal solution S. Therefore, the assumption that there exists more than one fractional transportation period between two full inventory periods is false.

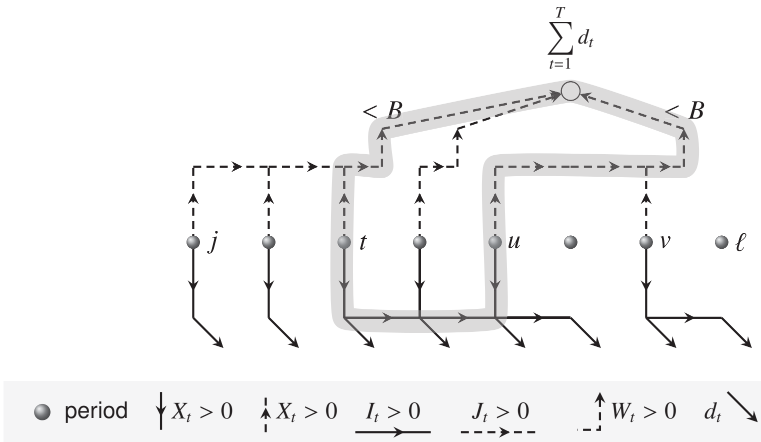

Figure 5: ULS-B^{const} with transportation periods between two full inventory periods $j-1, \ell \in \{1, 2, \ldots, T\}$: *Network flow representation*

□

6.2. Dynamic programming algorithm

In the light of Property 3, the optimal solution of ULS-B^{const} can be calculated by decomposing the problem into blocks of periods delimited by full inventory periods.

Definition 8 (Block). *The set of time periods $\{j, j+1, \ldots, \ell\}$ between two consecutive full inventory periods $j-1$ and ℓ is called a* **block**, *where $j < \ell$, $\forall j, \ell \in \{2, 3, \ldots, T\}$.*

Under Property 3, let us define a block via three periods $[j, \ell]_f$:

- two consecutive full inventory periods $j-1$ and ℓ, i.e. an initial period j with zero entering stocks and a final period ℓ with zero ending stocks;
- a fractional transportation period f, with $j \leq f \leq \ell$.

Between two full inventory periods, there is at most one fractional transportation period, all other transportation periods are full transportation periods. Hence, the total production is equal to the total demand between j and ℓ, i.e. $d_{j\ell} = \sum_{t=j}^{\ell} d_t$. Given that $\alpha = 1$, the total generated by-product is also equal to $d_{j\ell}$.

The number of full transportation periods inside a block $[j, \ell]_f$ is equal to $\left\lfloor \frac{d_{j\ell}}{B} \right\rfloor$. Let K denote the total number of transportation periods, full and fractional, inside a block $[j, \ell]_f$, i.e.:

$$K = \left\lceil \frac{d_{j\ell}}{B} \right\rceil.$$

Let Δ be the quantity transported during the fractional transportation period:

$$\Delta = d_{j\ell} - \left\lfloor \frac{d_{j\ell}}{B} \right\rfloor B.$$

There is no fractional transportation period between j and ℓ, if $\Delta = 0$. By convention, f is set to zero, when $\Delta = 0$.

Definition 9 (Sub-Block)**.** *Let $[j, \ell]_f$ be a block. The set of periods $\{s, s+1, \ldots, t\}$, defined by:*

- *$s - 1$ and t: two consecutive transportation periods between j and ℓ, $j \leq s \leq t \leq \ell$;*
- *$n \in \{0, 1, \ldots, K\}$: number of transportation periods performed before period s since period j.*

*is called **sub-block** and is denoted by $[s, t]_f^{jn}$.*

By virtue of Definition 9, the following information is known for each sub-block of type $[s, t]_f^{jn}$:

- The entering inventory level at period s and the ending inventory level at period t of the by-product are null, since transportation occurs by definition at periods $s - 1$ and t:

$$J_{s-1} = J_t = 0 \tag{11}$$

- The entering inventory level of the main product in period t equals to:

$$I_{s-1} = \begin{cases} nB & -d_{j,s-1}, \text{ if } f \geq s \text{ or } f = 0 \\ (n-1)B + \Delta & -d_{j,s-1}, \text{ if } f < s \end{cases} \tag{12}$$

- The ending inventory level of the main product at period t can be calculated as follows:

$$I_t = \begin{cases} (n+1)B & -d_{jt}, \text{ if } f > t \text{ or } f = 0 \\ nB + \Delta & -d_{jt}, \text{ if } f \leq t \end{cases} \tag{13}$$

The cost of sub-block $[s, t]_f^{jn}$ can be evaluated by solving a classical single-item lot-sizing problem using an $O(T \log T)$ dynamic programming algorithm (see e.g. Federgruen and Tzur (1991); Wagelmans et al. (1992); Aggarwal and Park (1993)) as follows:

- Calculate the entering and ending inventory levels of the main products via expressions (12)-(13).

- Pre-treat the inventories of the main product, in order to obtain zero entering and zero ending inventories of the main product.
- Update the production cost at each period i by integrating the inventory costs of the by-product, as follows: $p_i := p_i + \sum_{j=i}^{t-1} \hat{h}_j,\ s \leq i \leq t$.
- Add the cost of the pretreated initial stock of the main product to this cost.

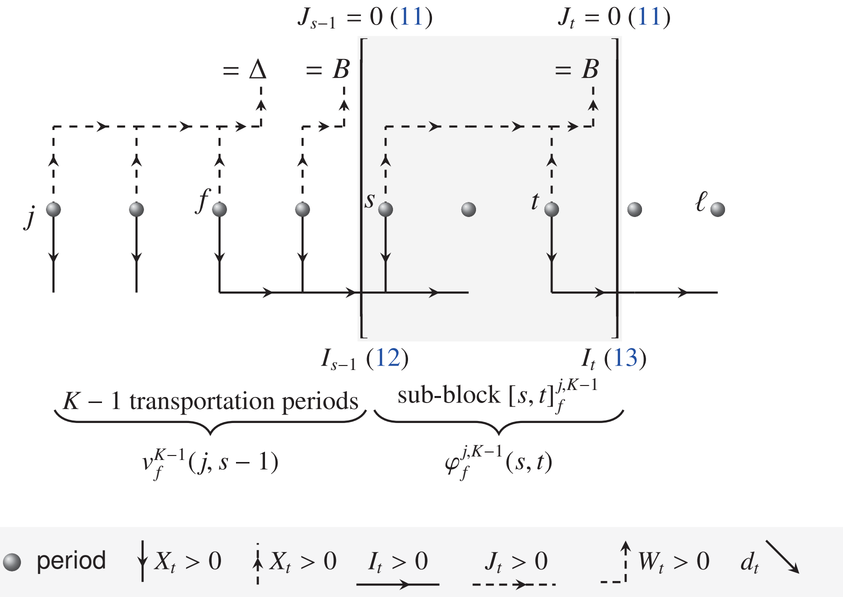

Figure 6: Inside a block $[j,\ell]_f$ with K transportation periods: *Network flow representation*

Denote by $\varphi_f^{jn}(s,t)$ the optimal value of sub-block $[s,t]_f^{jn}$. Figure 6 gives an example of a decomposition into sub-blocks. Let f be the fractional transportation period, the optimal cost $v_f^K(j,t)$ from period j to a given transportation period t including n transportation periods is provided by:

$$v_f^n(j,t) = \min_{j \leq s \leq t} \left\{ v_f^{n-1}(j, s-1) + \varphi_f^{j,n-1}(s,t) \right\} + g_t, \tag{14}$$

with $v_f^{n-1}(j, j-1) = 0$.

The optimal solution of a block of type $[j,\ell]_f$ is given by $v_f(j,\ell)$:

$$v_f(j,\ell) = \min_{j+K-1 \leq s \leq \ell} \left\{ v_f^K(j,s) + \sum_{t=s}^{\ell-1} h_t d_{t+1,\ell} \right\}$$

The cost of an optimal solution between two full inventory periods $j-1$ and ℓ is the minimum value among all possible costs $v_f(j,\ell)$ between two consecutive full inventory periods $j-1$ and ℓ, for all $f \in \{j, j+1, \ldots, \ell\}$:

$$v(j,\ell) = \begin{cases} \min_{j \leq f \leq \ell} v_f(j,\ell), & \text{if } \Delta > 0 \\ v_0(j,\ell), & \text{if } \Delta = 0 \end{cases}$$

Proposition 3. *For initial null inventory levels of the main product and by-product, the optimal cost of the ULS-B problem is equal to $C(T)$ given by the last step of the following algorithm, which proceeds forward in time from period 1 to T, $\forall \ell \in \{1, 2, \ldots, T\}$:*

$$C(0) = 0 \tag{15}$$

$$C(\ell) = \min_{1 \leq j \leq \ell} \left\{ C(j-1) + v(j, \ell) \right\} \tag{16}$$

Proof. Under Property 3, an optimal solution of ULS-B^{const} can be partitioned in a series of blocks delimited by consecutive full inventory periods. Given a period ℓ, the cost of ULS-B^{const} over the periods from 1 to ℓ can be recursively expressed via the minimum for all $j \in \{1, 2, \ldots, \ell\}$ of the sum of: **(i)** the cost up to the previous full inventory period $C(j-1)$, and **(ii)** the cost of the block between two full inventory periods $j-1$ and ℓ.

Recursion formula (16) performs the exhaustive evaluation of all possible decompositions into blocks of full inventory periods. By induction, it follows that $C(T)$ gives the optimal cost of the ULS-B^{const} problem. □

Proposition 4. *The optimal value $C(T)$ can be computed in $O(T^6 \log T)$.*

Proof. The complexity of the dynamic programming algorithm given in Proposition 3 is due to the calculation of the cost of sub-blocks $\varphi_f^{j,n-1}(s, t)$. Each sub-block $[s, t]_k^{jn}$ can be evaluated by solving a classical single-item lot-sizing problem using an $O(T \log T)$ dynamic programming algorithm (see e.g. Federgruen and Tzur (1991); Wagelmans et al. (1992); Aggarwal and Park (1993)). Hence, the calculation of cost $v_f^K(j, t)$ is done in $O(T^2 logT)$ for any given periods $f, j, t \in \{1, 2, \ldots, T\}$. Given the composition of functions $v_f^K(\bullet, \bullet)$, $v_f(\bullet, \bullet)$, $v(\bullet, \bullet)$ and $C(\bullet)$, it follows that the optimal cost of the ULS-B^{const} $C(T)$ can be computed in $O(T^6 \log T)$.

□

7. Conclusion and perspectives

Inspired by the circular economy thinking, this paper introduced and investigated a new version of the lot-sizing problem. The addressed problem is an extension of the classical single-item uncapacitated lot-sizing problem, which integrates explicitly the management of a by-product storable in limited non-decreasing quantities. We showed that ULS-B problem with time-dependent inventory capacities of the generated by-product is weakly $\mathcal{NP}$-Hard, and provided a pseudo-polynomial dynamic programming algorithm based on the solution decomposition into blocks. For the special case of ULS-B with stationary inventory capacities of the generated by-product, we gave a polynomial dynamic programming algorithm based on the derived structural properties of the optimal solutions.

The studied generic ULS-B problem is a first step towards the implementation of industrial symbiosis networks. Considering such a relation between two or several industries, gives rise to multiple questions: How to better integrate real-life features (proximity between industries, by-product storability, inventory capacities, etc.)? How to coordinate the joint production of multiple industries? How to regulate and formalize the agreements specifying the complementary relationship between industries? As a future research, it will be interesting to model these new production planning problems, to study the induced complexity and to propose efficient solution methods.

Notes

1. The directive 2008/98/EC of the European Parliament and of the Council of 19 November 2008 on waste and repealing certain Directives. ELI: `https://eur-lex.europa.eu/eli/dir/2008/98/oj`

2. The communication from the Commission to the Council and the European Parliament on the Interpretative Communication on waste and by-products, number 52007DC0059: https://eur-lex.europa.eu/legal-content/EN/TXT/?uri=CELEX%3A52007DC0059

3. The communication from the Commission to the European Parliament, the Council, the European Economic and Social Committee and the Comittee of the Regions. Closing the loop - An EU action plan for the Circular Economy: http://eur-lex.europa.eu/legal-content/EN/TXT/?uri=CELEX:52015DC0614

4. PIICTO: https://piicto.fr

5. Kalundborg Symbiosis: http://www.symbiosis.dk

6. Deltalinqs: https://www.deltalinqs.nl

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