Quantization applied to the visualization of low-probability flooding events

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Content

Hydraulic simulators

Input variables

Random natural phenomena
Breaches

$\Rightarrow$

Computer Code

Black Box

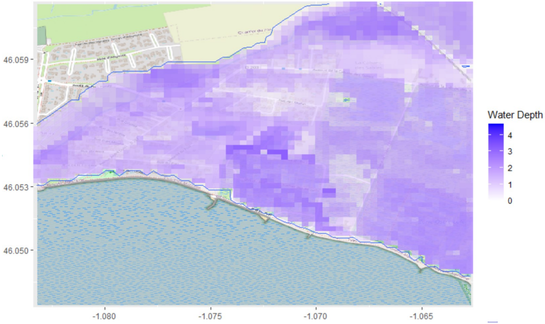
$\Rightarrow$

Output

$Y = [\max_t(Water\ Depth_{i,j})]_{i,j \in grid}$

One simulation $\sim$ several hours

Output : Flooding map



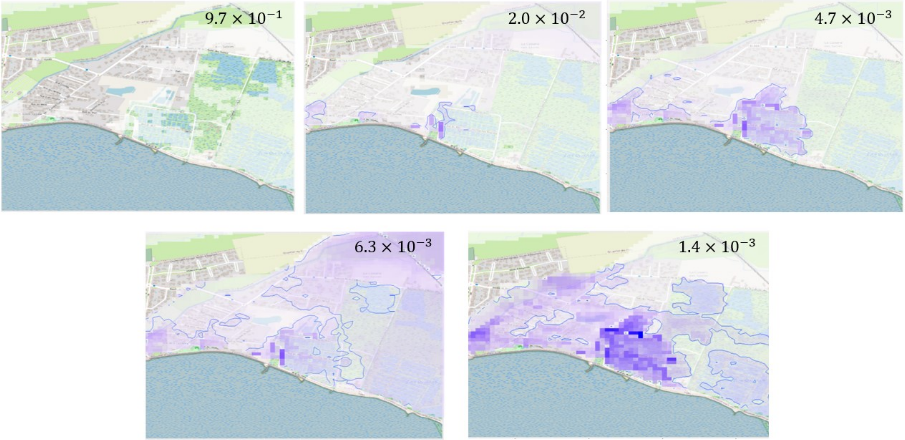
Visualization problem

Flooding map : easy to understand for everyone (Urban Planners, Decision makers)

But how to show a set of flooding maps that best represent the probability law associated to the flooding event ?

$\Rightarrow$ K-Means Clustering : Identify K Voronoi cells such that the mean distance between an observation and its nearest cell centroid is minimized

Objective


9.7×10^{-1}
2.0×10^{-2}
4.7×10^{-3}
6.3×10^{-3}
1.4×10^{-3}

Problem formulation I

Input space $= \chi =$ Natural Phenoma parameters $\times$ Breach parameters

Output space $= \mathcal{Y}$ the space of pixelated maps, with inner product $\langle ., . \rangle_{\mathcal{Y}}$

We introduce the random field Y :

$$\begin{aligned} Y \colon D &\to \mathcal{Y}, \langle ., . \rangle_{\mathcal{Y}} \\ x &\mapsto Y(x). \end{aligned}$$

Problem formulation II

Quantization problem : Find for a given $\ell \in \mathbb{N}$, $\Gamma = \{\gamma_1, \gamma_2, ..., \gamma_\ell\} \in \mathcal{E}^\ell$ ℓ representatives of $Y(X)$

Closest representative map function:

$$q_\Gamma : \mathcal{Y} \to \Gamma$$
$$y \mapsto q_\Gamma(y) = \underset{\gamma_i \in \Gamma}{\arg\min} \|y - \gamma_i\|_{\mathcal{Y}}$$

.

Quantization error: $e(\Gamma) = \left[\mathbb{E}\left[\|Y(X) - q_\Gamma(Y(X))\|^2_{\mathcal{Y}}\right]\right]^{\frac{1}{2}}$

Objective: Find

$$\Gamma^\star = \{\gamma_1^\star, .., \gamma_\ell^\star\} = \underset{\Gamma \in \mathcal{Y}^\ell}{\arg\min}\,(e(\Gamma))$$
$$= \underset{\Gamma \in \mathcal{Y}^\ell}{\arg\min} \left[\mathbb{E}\left[\min_{i \in \{1..\ell\}} \|Y(X) - \gamma_i\|^2_{\mathcal{Y}}\right]\right]^{\frac{1}{2}}$$

Key points

Quantization is performed in a specific context :

1. Expensive-to-evaluate simulators : metamodels adapted to spatial output
2. Low probability event : standard Monte Carlo sampling approach inefficient
3. Quantization in a space of maps (Adapted metamodel, storage)

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Theorem Kieffer, Cuesta-Albertos

Theorem

If

- $\mathcal{Y}$ *is of finite dimension* q
- $\forall z, y \in \mathcal{Y}, \langle z, y\rangle_{\mathcal{Y}} = \sum_{i=1}^{n} \lambda_i z_i y_i$ *with* $\forall i, \lambda_i > 0$
- $\mathbb{E}\left[\|Y(X)\|_{\mathcal{Y}}^2\right] < +\infty$

then $\forall i \in \{1 \dots \ell\}, \mathbb{E}\left[Y(X) \mid Y(X) \in C_i^{\Gamma^\star}\right] = \gamma_i^\star$

by defining C_j^Γ the Voronoi cells associated with a quantization Γ : $C_j^\Gamma = \{y \in \mathcal{Y}, q_\Gamma(y) = \gamma_j\}$.

This means that the representatives of an optimal quantification coincide with the cells centroids.

Lloyd's algorithm

Algorithm Lloyd's algorithm

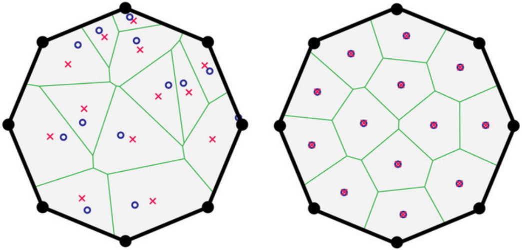
$\Gamma^{[0]} \leftarrow \{\gamma_0^{[0]}, \dots, \gamma_\ell^{[0]}\}$

1: **while** not stopping criterion **do**

$\forall j \in \{1, \dots, \ell\}, \gamma_j^{[k+1]} \leftarrow \mathbb{E}\left[Y(X) \mid Y(X) \in C_j^{\Gamma^{[k]}}\right]$

$k \leftarrow k + 1$

2: **end while**



Lloyd in our case

The main point is to compute at each iteration the conditional expectation $\mathbb{E}\left[Y(X) \mid Y(X) \in C_j^{\Gamma^{[k]}}\right]$

Problem here : In the flooding case, one prevailing Voronoi cell of empty maps (ie without water).

$\Rightarrow$ Monte Carlo sampling techniques not adapted for other clusters

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Principle of Importance Sampling

Objective : Estimate $\mathbb{E}\left[g(Y(X))\right]$ with $g : \mathcal{Y} \to \mathbb{R}^p$ such as $\mathbb{E}\left[g(Y(X))^2\right] < +\infty$

Main idea : The representation of $\mathbb{E}\left[g(Y(X))\right]$ as an expectation is not unique :

$$\mathbb{E}\left[g(Y(X))\right] = \mathbb{E}\left[g(Y(\tilde{X}))\frac{f_X(\tilde{X})}{\nu(\tilde{X})}\right]$$

with $\tilde{X}$ a random variable with density function ν with $supp(f_X) \subset supp(\nu)$

Estimator with importance sampling

From this last representation :

$$\hat{E}_n^{IS} = \frac{1}{n} \sum_{k=1}^{n} g(Y(\tilde{X}^{(k)})) \frac{f_X(\tilde{X}^{(k)})}{\nu(\tilde{X}_k)}$$

with $(\tilde{X}^{(k)})_{k=1}^{n}$ be a n-sample of $\tilde{X}$

Its covariance matrix is : $\mathbb{V}(\hat{E}_n^{IS}) = \frac{1}{n}\mathbb{V}(g(Y(\tilde{X}))\frac{f_X(\tilde{X})}{\nu(\tilde{X})})$

In comparison to $\frac{1}{n}\mathbb{V}(g(Y(X)))$ in a classical MC

Idea : Choose ν that minimises variance

Importance sampling combined with quantization

$$\mathbb{E}\left[Y(X) \mid Y(X) \in C_j^\Gamma\right] = \frac{\mathbb{E}\left[Y(X)\mathbb{1}_{Y(X)\in C_j^\Gamma}\right]}{\mathbb{E}\left[\mathbb{1}_{Y(X)\in C_j^\Gamma}\right]}$$

And an estimator of $\mathbb{E}\left[Y(X) \mid Y(X) \in C_j^\Gamma\right]$:

$$\hat{E}_n^{IS}(\Gamma, j) = \frac{\frac{1}{n}\sum_{k=1}^n Y(\tilde{X}^{(k)})\mathbb{1}_{Y(\tilde{X}^{(k)})\in C_j^\Gamma}\frac{f_X(\tilde{X}^{(k)})}{\nu(\tilde{X}_k)}}{\frac{1}{n}\sum_{k=1}^n \mathbb{1}_{Y(\tilde{X}^{(k)})\in C_j^\Gamma}\frac{f_X(\tilde{X}^{(k)})}{\nu(\tilde{X}_k)}}$$

Density function ν : Uniform distribution

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Method [Perrin et al, 2021]

1. FPCA : Write every map $Y(x)$ as a linear combination of n_{pc} maps :
$$Y(x) = t_1(x)Y_1^{\text{pca}} + \cdots + t_{n_{pc}}(x)Y^{\text{pca}_{n_{pc}}}$$
2. Gaussian process regression on every axis to predict $(\hat{t}_1(x^\star), \ldots, \hat{t}_{n_{pc}}(x^\star))$ for a new $x^\star$

$\Longrightarrow$ Work with a large sample of predicted maps $\hat{Y}(\tilde{X}^{(k)})_{k=1}^{n}$

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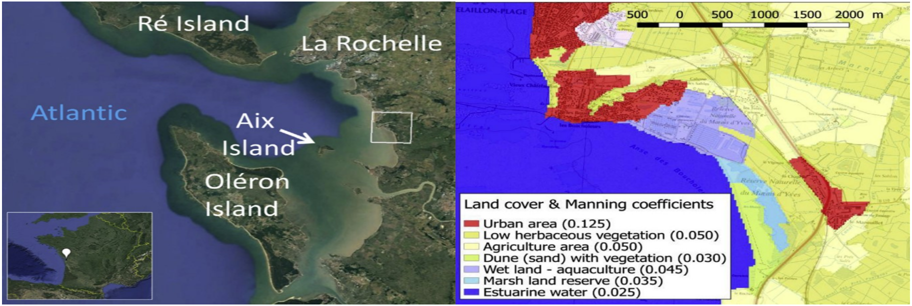
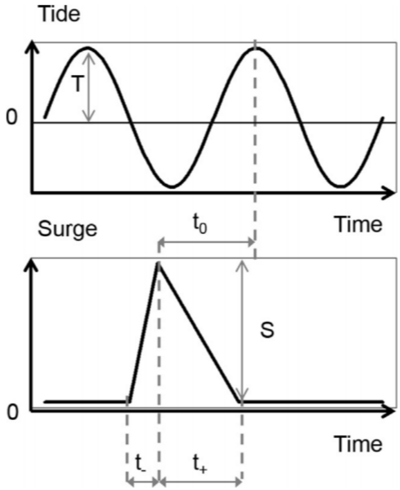
Les Boucholeurs

- French Atlantic Coast near La Rochelle
- Hit by the storm Xynthia (27-28 February 2010)



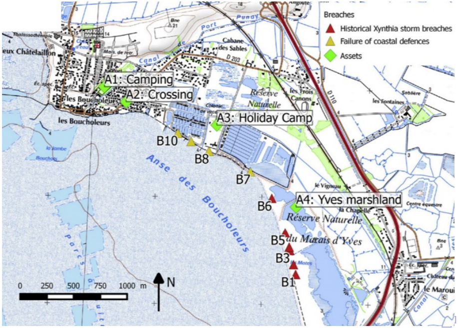
Natural phenomena variables

- High-tide level
- Surge peak amplitude
- Phase difference
- Time duration of the rising part
- Time duration of the falling part

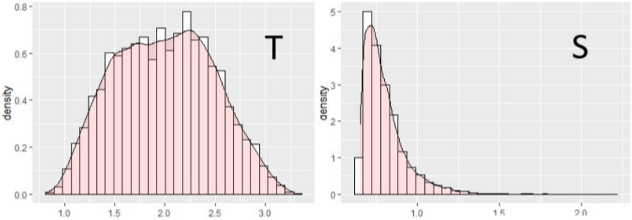


Breach variables

1. Location of the breaches : 10 different locations
   - 6 natural protections based on historical observations
   - 4 artificial dykes near vulnerable zones
2. Erosion rate : topographic level after failure as a fraction of initial crest level



Density function



Offshore conditions : Historical observations

Important offset added to the Surge that models the rise of the sea level

Breach variables :

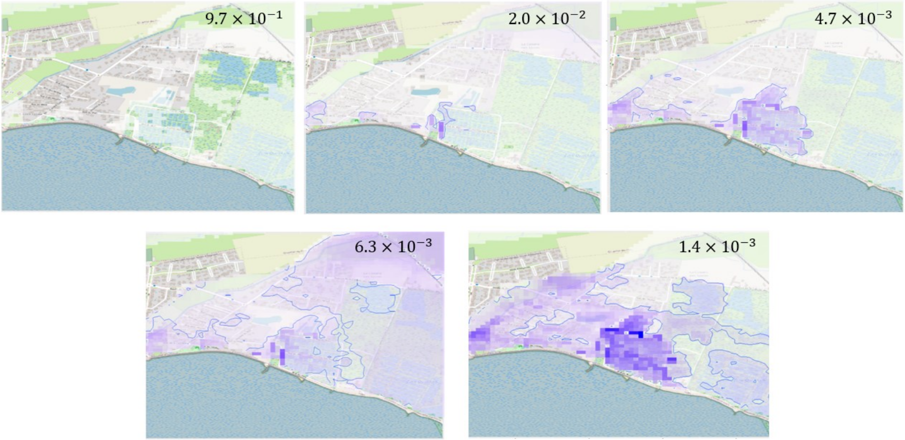
- Probability of failure : p_{fail} function of $T, S, t0, t_-, t_+$
- If failure : Uniform probability law

Data

1300 flood maps simulated as follows :

- Natural phenomena variables sampled as the beginning of the Sobol Sequence
- 500 simulations without breach
- 800 breaches simulated uniformly ($\mathcal{U}_{\{1,..,10\}}$ for the breach location and $\mathcal{U}_{[0,1]}$ for the erosion rate)

Results



Future work

- Adapt the number of prototype maps
- Biaised density ν
- Add a study of the input space