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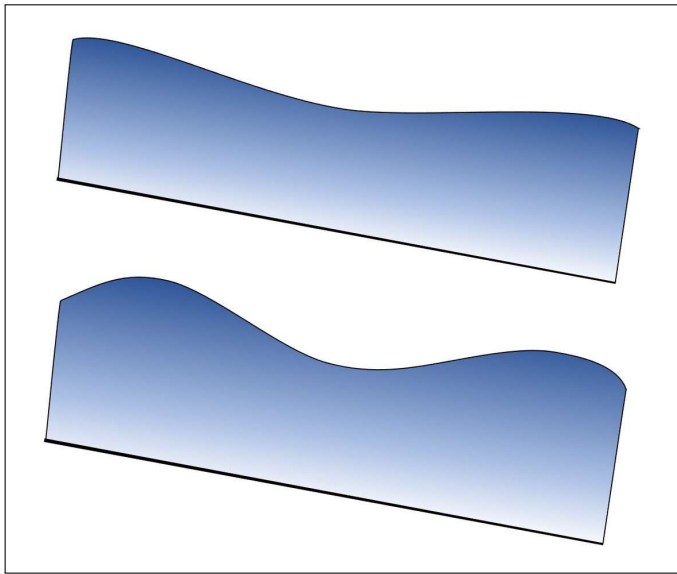
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Graphical Abstract

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Highlights

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- We introduced small density variations within a gravity–driven falling film via a barotropic equation of state and investigated how this affects the flow temporal linear stability.
- In the final depth–averaged evolution equations compressibility is reflected in two additional second–order terms.
- A weak compressibility boosts the onset of interfacial instability, especially in low–inertial regimes and along modest slopes.
- We detected an extra flow rate of hydrostatic origin due to compressibility and complemented our analysis with the wave–hierarchy theory.

Compressibility–induced destabilisation of falling liquid films: an integral approach

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Abstract

We revisit the classical 2D problem of a gravity–driven liquid layer down an inclined plate (Kapitza, *Zh. Eksp. Teor. Fiz.*, vol. 18 (1), 1948, pp. 3–28), relaxing the usual assumption of homogeneous fluid. We set out to answer three major issues. When the fluid density is allowed to vary, (i) how does this feature structurally affect the formulation of a low–dimensional depth–averaged model? (ii) To what extent and (iii) by virtue of which physical mechanism does compressibility participate in the long–wave interfacial instability? To provide the relevant answers, (i) we first make use of a second–order asymptotic expansion in the shallowness parameter to develop a weakly–compressible boundary–layer system: starting from a two–equation momentum–integrated model, an additional barotropic equation of state is required for closure purposes. In this respect, (ii) a temporal linear stability analysis is performed: it is revealed that compressibility plays a destabilising role whose magnitude is enhanced at intermediately tilted configurations,

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and the more the Reynolds number approaches the critical threshold in the incompressible limit. (iii) We finally interpret the ensuing dispersion relation under the convenient framework of two-wave hierarchy, initiated by Whitham (*Linear and Nonlinear Waves*, Wiley—Interscience, 1974): the primary instability gets promoted by the flow compressibility as it contributes to deceleration of dynamic waves most significantly in the low-inertia regime. Indeed, compressibility locally acts as a further boost to the inertia-based mechanism of Kapitza instability by amplifying flow-rate variations within the liquid film.

Keywords: falling liquid films, interfacial instability, low-dimensional models

PACS: 47.15.Cb, 47.20.Ma, 68.15.+e

2000 MSC: 76E17, 76E19, 76M45

1. Introduction

Liquid layers sliding down an incline are routinely encountered in nature and represent a cross-disciplinary and highly topical object of study. Starting with the pioneer studies of Kapitza father-son team (Kapitza, 1948; Kapitza and Kapitza, 1949), visual observations have revealed the development of a wide variety of intriguing patterns along the fluid interface, from simple sinusoidal perturbations to strongly non-periodic three-dimensional solitons (Chang, 1994; Alekseenko et al., 1994).

This issue is also of practical relevance in many biological and industrial processes (Craster and Matar, 2009). Cooling towers, distillation units, multi-phase heat exchangers, fluid-phase separators, jet-film devices, power

12 station condenser tubes, absorption columns, electrolytic cells, scrubbers for
13 pollution abatement, injection systems for enhanced oil recovery, *etc.* all ben-
14 efit from the strong effect of superficial waves on the underlying processes of
15 heat and mass transfer. For instance, according to data reported by Frisk
16 and Davis (1972), the heat transfer intensification by waves forming along
17 a water film in presence of a co-current air flow attains more than 100 %
18 with respect to the flat-film scenario. On the other hand, for some appli-
19 cations such as coating operations, a uniform flow thickness is required and
20 instabilities should be prevented (Weinstein and Ruschak, 2004).

21 So far, the majority of works on wavy falling films is performed assuming
22 flow incompressibility, *i.e.* the density of a fluid element remains uniform and
23 constant. However, in many fields of science and engineering, this assumption
24 may constitute an oversimplification of the physical problem, possibly leading
25 to inaccurate conclusions. One example is the transport of carbon dioxide in
26 pipelines from the energy plants to the injection sites for CCUS applications.
27 When supercritical carbon dioxide is employed as solvent or carrier, in fact,
28 density turns out to be an essential parameter in determining the performance
29 of such a technological process. In this context, avoiding the unbounded
30 growth of superficial disturbances, which can result in the emergence of slugs
31 or even structural damages (Lu et al., 2020), is necessary for the safety of
32 the transport infrastructure.

33 Although the convective long-wave interfacial mode known as Kapitza
34 instability (Kapitza, 1948) constitutes a long-standing knowledge in case of
35 a tilted or vertical plate, a deep understanding of how density variations
36 enter this paradigm is still lacking in the literature. Thus, the link between

37 the compressibility and the occurrence of the Kapitza instability needs to
38 be clarified and has prompted us to address the following question: which
39 are the main implications of density inhomogeneities on the onset of Kapitza
40 instability in inclined falling liquid films?

41 Recently, the relevance of wavy film flows has led to a number of at-
42 tempts to achieve models for the evolution of the film thickness and its mean
43 velocity (or flow rate) and to find a compromise between the accuracy and
44 the computational effort. In most cases, the flow description is not too far
45 from its wavy-less configuration, designed as Nusselt state (Nusselt, 1916).
46 This makes the long-wave asymptotic expansion a feasible approach, which
47 forms the cornerstone of many models derived after the influential paper
48 of Benney (1966), who developed an evolution equation for the film height
49 by introducing a small-scale parameter. However, Benney's equation suffers
50 of finite-time blow-up of the time-dependent solution. This problem was
51 addressed by Shkadov (1967) assuming that streamwise variations are small
52 as compared with those developing in the crosswise direction, and through
53 pressure removal boundary layer equations (BLEs) ensue. These are then av-
54 eraged over the fluid depth to capture the main physical features of the flow
55 by means of integral variables. Nonetheless, Shkadov's system of equations
56 fails in capturing the correct long-wave instability threshold. This issues
57 was addressed by Ruyer-Quil and Manneville (1998, 2000), who introduced
58 the weighted residual integral boundary-layer model (WRIBL) and assured
59 model consistency by formulating a closure law for the wall shear stress.

60 In this paper, our purpose is to deal with a weakly inhomogeneous medium
61 to investigate whether and how the action of a low compressibility enhances

62 or mitigates the onset of long-wave interfacial instability. We therefore start
63 by applying Benney’s modelling strategy to a barotropic flow in a weakly-
64 compressible scenario. We globally characterise it in terms of compressibility
65 by means of the Mach number and formulate a coupled system of two evo-
66 lution equations by making use of the depth-wise averaging method based
67 on the classical long-wave expansion as in Lavalle et al. (2015, 2017), *i.e.* by
68 integration of the momentum balance (momentum integral method or MIM).
69 The resulting model is comprehensive of second-order viscous diffusion ef-
70 fects, which allow us to achieve good agreement in the incompressible limit in
71 terms of the cut-off wavenumber with the Orr-Sommerfeld solution (Kalli-
72 adasis et al., 2013).

73 Our study focuses on the influence of compressibility on the development
74 of linear surface waves on a liquid film falling down an inclined wall under
75 a shear-free atmosphere (figure 1). For this, we consider the primary insta-
76 bility of the weakly-compressible uniform base flow and solve the temporal
77 stability problem based on the long-wave model equations. By doing this,
78 we answer two additional questions: (i) how does compressibility affect the
79 formulation of depth-integrated equations? (ii) Which physical mechanism
80 does the compressibility trigger in the long-wave interfacial instability?

81 Behind the usual incompressible way of modelling falling liquid layers,
82 it is assumed that the speed of sound, when compared to the convective
83 velocity scale, is sufficiently high to be considered infinite. Therefore, (i)
84 a unique velocity scale appears in the problem and (ii) the fluid density is
85 uniform and constant. On the contrary, when a finite speed of sound is taken
86 into account, the scenario significantly changes. (i) Convective transport and

87 pressure wave propagation occur at disproportional rates, thereby requiring a
88 proper incorporation of an additional dimensionless group in the problem. In
89 this regard, the Sarrau–Mach number can be used to express the magnitude
90 of the fluid speed as compared to the sound speed within the same medium.
91 In addition, (ii) the fact that density field is allowed to vary in space and
92 time demands the introduction of an Equation of State (EoS) among the
93 governing equations.

94 Unfortunately, very little attention, to the best of our knowledge, has
95 been up to now devoted to the assessment of the impact of compressibility
96 on the film long–wave instability. In fact, only a few works tried to tackle
97 this issue.

98 An extension of long–wave models to weakly–compressible barotropic
99 flows is first proposed by Richard (2021). Compressibility–related effects
100 are captured by means of a dedicated Mach number, defined by means of
101 the incompressible surface waves celerity, and, in the limit where the sound
102 speed goes to infinity, the incompressible version of the model is correctly
103 recovered. However, the system of four Favre–averaged equations derived
104 by Richard (2021) is intended for simulation of coastal waves and the author
105 frames his argumentation around the ultimate goal of correctly predicting
106 tsunamis’ arrival time. Although the long–wave assumption still holds for a
107 tidal wave in a deep ocean, the relevant spatial scales involved widely differs
108 from the ones we are interested in. Moreover, in Richard (2021), the wave
109 propagation is studied within an inviscid medium, neglecting viscous effects.

110 Such friction terms have also been neglected in the work of Bresch et al.
111 (2020), who developed an augmented skew–symmetric system of depth–

112 integral equations with capillarity. Their work aims at ensuring the stability
113 of numerical schemes in presence of large gradients of fluid height or fluid
114 density.

115 In the context of flows within a narrow interstice formed between two
116 surfaces, Almqvist et al. (2019) consider a class of iso-viscous fluids obeying
117 a constitutive power-law density-pressure relationship. Lubrication theory,
118 scaling and asymptotic analysis are extensively used in that work to show
119 that the degree of compressibility for a thin film flow determines whether
120 the terms governing inertia may or may not be neglected. Notwithstanding
121 the rigorousness of their procedure, the study of a capillary flow is not at all
122 comparable to a free-surface gravity-driven liquid film.

123 We conclude by recalling the fundamental results regarding the linear
124 stability problem of a falling liquid film in a passive gas or shear-free at-
125 mosphere, which is the configuration studied in this work. Benjamin (1957)
126 and Yih (1963) solved the temporal linear stability problem formulated by Orr
127 (1907) and Sommerfeld (1908) in the context of a gravity-driven incompress-
128 ible film flow. In particular, they detected the long-wave instability threshold
129 in terms of a critical Reynolds number $Re_{cr} = 5/6 \cot \beta$, where β identifies
130 the inclination angle, being the Reynolds number based on the mean film
131 flow velocity. Their analysis reveals that inertia destabilizes long waves and
132 the related mechanism has been explained either through the shift between
133 the vorticity perturbation and the perturbed interface (Kelly et al., 1989;
134 Kalliadasis et al., 2013; Smith, 1990), or via the time lag at which flow rate
135 adapts to its inertialess target value (Dietze, 2016). With the aim to inves-
136 tigate the role of compressibility on the long-wave instability, we follow the

137 latter approach by considering the effect of compressibility on the inertialess
138 flow rate, similarly to Lavallo et al. (2019), who applied the same methodol-
139 ogy to explain the confinement-induced stabilisation of falling liquid films.
140 Finally, we complement this analysis by studying the role of compressibility
141 via the two-wave competition theory formulated by Whitham (Whitham,
142 1974), and employed by Samanta et al. (2011) and Samanta (2014) for liquid
143 films down a slippery inclined plane or for shear-imposed falling films.

144 Accordingly, the structure of our paper is as follows. Section § 2 contains
145 the basic governing equations, the boundary conditions of the problem, and
146 the definition of the principal dimensionless groups, together with the long-
147 wave scaling. Then, the low-dimensional modelling is discussed in § 3, from
148 the specification of the EoS to the derivation of the weakly-compressible
149 integral model. This will serve in the second part of the manuscript, devoted
150 to the linear temporal stability eigen-problem, whose compatibility yields the
151 dispersion relation outlined in § 4. To follow, § 5 presents our main findings
152 in terms of critical threshold and parametric study of celerity branches. The
153 mechanism governing the influence of compressibility on the film stability is
154 finally elucidated in § 6. Concluding remarks are summarised in § 7, while
155 some details of the analysis that were not included in the main body of the
156 text are given in the appendix for completeness.

157 **2. Flow configuration and theoretical formulation**

158 Herein we consider the two-dimensional compressible flow of a gravity-
159 driven iso-viscous liquid film, falling along a tilted wall within a shear-
160 free atmosphere, as sketched in figure 1. The liquid film is Newtonian.

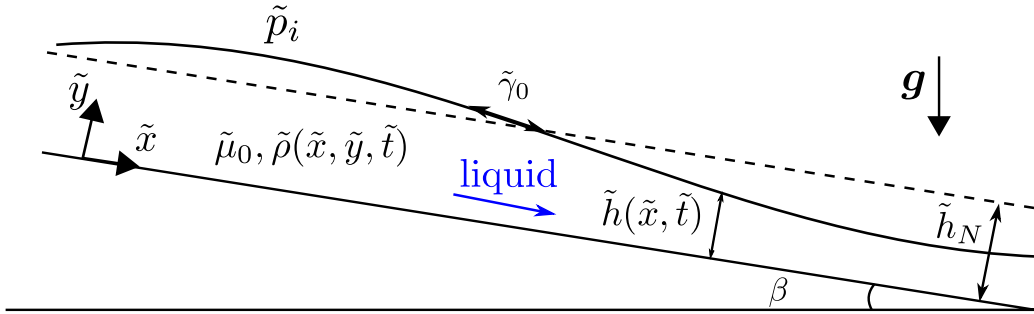


Figure 1: Schematic diagram of the 2D slightly compressible flow of a wavy gravity-driven liquid film with uniform and constant viscosity $\tilde{\mu}_0$ and surface tension $\tilde{\gamma}_0$, exhibiting a non-uniform and variable density $\tilde{\rho}(\tilde{x}, \tilde{y}, \tilde{t})$. The coordinate system is defined by $\langle \tilde{x}, \tilde{y} \rangle$. The fluid layer, of variable thickness $\tilde{h}(\tilde{x}, \tilde{t})$, flows under the action of gravity \mathbf{g} along a plate having an inclination angle β with respect to the horizontal direction. \tilde{h}_N refers to Nusselt solution (Nusselt, 1916) and denotes the waveless film thickness. An interfacial constant and uniform normal pressure \tilde{p}_i is present.

161 $\beta \in]0, \frac{\pi}{2}]$ refers to the angle of inclination formed between the wall and the
 162 horizontal direction. The Cartesian coordinate axes \tilde{x} and \tilde{y} are placed along
 163 the streamwise and crosswise flow directions, respectively, being the origin
 164 of the spatial reference frame located at the wall; $\tilde{t} \in \mathbb{R}_0^+$ specifies the time
 165 coordinate. Assume that, with the exception of density $\tilde{\rho} \in \mathbb{R}^+$, the physi-
 166 cal properties of the liquid, such as dynamic viscosity $\tilde{\mu}_0 \in \mathbb{R}^+$ and surface
 167 tension $\tilde{\gamma}_0 \in \mathbb{R}^+$, are uniform within the physical fluid domain $\tilde{\Psi}$, defined as

$$\tilde{\Psi}(\tilde{t}) = \left\{ (\tilde{x}, \tilde{y}) \in \tilde{\mathbb{R}}^2 \mid 0 \leq \tilde{y} \leq \tilde{h}(\tilde{x}, \tilde{t}) \right\}, \quad (1)$$

168 where \tilde{h} is a dimensional function tracing the spatial and temporal evolution
 169 of the wavy film free surface.

170 *2.1. Governing equations*

171 At the continuum level, the dimensional form of the governing equations
 172 of motion enforcing the conservation of mass and momentum for the com-
 173 pressible flow of the Newtonian falling film reads:

$$\partial_{\tilde{t}} \tilde{\rho} + \tilde{\nabla} \cdot (\tilde{\rho} \tilde{\mathbf{v}}) = 0 \quad (2a)$$

$$\tilde{\rho} \left(\partial_{\tilde{t}} \tilde{\mathbf{v}} + (\tilde{\mathbf{v}} \cdot \tilde{\nabla}) \tilde{\mathbf{v}} \right) = -\tilde{\nabla} \tilde{p} + \tilde{\rho} \mathbf{g} + \tilde{\mu}_0 \left(\tilde{\nabla}^2 \tilde{\mathbf{v}} + \left(\frac{1}{3} + \vartheta \right) \tilde{\nabla} (\tilde{\nabla} \cdot \tilde{\mathbf{v}}) \right), \quad (2b)$$

174 where $\tilde{\mathbf{v}} = (\tilde{u}, \tilde{v})$ and \tilde{p} denote, respectively, the film velocity vector and
 175 the thermodynamic pressure, whereas $\mathbf{g} = (g \sin \beta, -g \cos \beta)$ is the grav-
 176 itational acceleration. The parameter labelled by $\vartheta = \tilde{\zeta}_0 / \tilde{\mu}_0$ expresses the
 177 ratio between the expansion viscosity $\tilde{\zeta}_0 \in \mathbb{R}$ and the dynamic viscosity
 178 $\tilde{\mu}_0$. Although ϑ is conventionally set to zero invoking Stokes' hypothesis
 179 ($\tilde{\zeta}_0 \equiv 0$) (Batchelor, 2000), we will not assume any particular value in or-
 180 der to preserve the widest possible generality throughout the paper. As will
 181 be demonstrated in § 3.2, this choice has no consequences in the ultimate
 182 formulation of the reduced model (25).

183 The flow system is subject to the following boundary conditions. At the
 184 rigid bottom $\tilde{y} = 0$, the no-slip and no-penetration conditions lead to

$$\tilde{\mathbf{v}}|_0 = \mathbf{0}. \quad (3)$$

185 At the free surface $\tilde{y} = \tilde{h}(\tilde{x}, \tilde{t})$, the balance of normal and tangential stress
 186 components for the shear-free film yields the dynamic coupling conditions

$$\left[\tilde{\mathbf{n}}^T \cdot \tilde{\mathbf{T}}^{(\tilde{\mathbf{n}})} \right] = \tilde{\gamma}_0 \tilde{\nabla} \cdot \tilde{\mathbf{n}} \quad (4a)$$

$$\left[\tilde{\mathbf{t}}^T \cdot \tilde{\mathbf{T}}^{(\tilde{\mathbf{n}})} \right] = 0, \quad (4b)$$

187 where $\tilde{\gamma}_0$ is the surface tension and $\tilde{\mathbf{T}}^{(\tilde{\mathbf{n}})}$ is the fluid stress vector at the
 188 interface, whose orientation is determined by

$$\tilde{\mathbf{n}} = \left\{ -\partial_{\tilde{x}}\tilde{h}, 1 \right\}^T / \sqrt{1 + \left(\partial_{\tilde{x}}\tilde{h}\right)^2} \quad (5a)$$

$$\tilde{\mathbf{t}} = \left\{ 1, \partial_{\tilde{x}}\tilde{h} \right\}^T / \sqrt{1 + \left(\partial_{\tilde{x}}\tilde{h}\right)^2} \quad (5b)$$

189 as normal and tangential unit column vector, respectively. Square brackets
 190 are used in (4) to designate the jump in any quantity of interest across the in-
 191 terface. Lastly, being the substantial derivative symbolised by
 192 $D(\star)/D\tilde{t} = \partial_{\tilde{t}}(\star) + \tilde{\mathbf{v}} \cdot (\tilde{\nabla}\star)$, an additional kinematic condition for the gas-
 193 liquid interface is introduced as follows

$$\frac{D}{D\tilde{t}} \left(\tilde{y} - \tilde{h}(\tilde{x}, \tilde{t}) \right) = 0. \quad (6)$$

194 2.2. Scaling and dimensionless formulation

195 To make the problem dimensionless, we choose the value of the density at
 196 the gas–fluid interface as the reference scale for density $\tilde{\rho}_0$ in line with Richard
 197 (2021). This scale is convenient since at $\tilde{y} = \tilde{h}$ the hydrostatic contribution
 198 on pressure and density fields is depth–independent. The Nusselt film thick-
 199 ness \tilde{h}_N (Nusselt, 1916) is chosen as the relevant length scale (figure 1),
 200 while we adopt the longitudinal characteristic speed as scale for the velocity
 201 $\tilde{U}_N = \tilde{q}_N/\tilde{h}_N = \tilde{\rho}_0 g \sin \beta \tilde{h}_N^2 / 3 \tilde{\mu}_0$, where \tilde{q}_N is the flow rate per unit of
 202 channel length:

$$\tilde{q}_N = \int_0^{\tilde{h}_N} \tilde{u}_N(\tilde{y}) d\tilde{y}, \quad (7)$$

203 being $\tilde{u}_N(\tilde{y})$ the well–known Nusselt parabolic velocity profile (Nusselt, 1916).

204 The average velocity \tilde{U}_N is indeed defined from the balance of the viscous
 205 friction force, $\propto \tilde{\mu}_0 \tilde{U}_N / \tilde{h}_N^2$, and the streamwise gravity force, $\propto \tilde{\rho}_0 g \sin \beta$.

206 The time and pressure scales are chosen as \tilde{h}_N/\tilde{U}_N and $\tilde{\rho}_0 \tilde{U}_N^2$, respectively (Lavalle
207 et al., 2015).

208 As a customary practice in the study of the wavy film dynamics, we will
209 adopt a shallow water approximation. Denoting by $\tilde{\mathcal{L}}$ a typical lengthwise
210 distance characterizing superficial corrugations, we define the following film
211 aspect ratio

$$\varepsilon = \frac{\tilde{h}_N}{\tilde{\mathcal{L}}} \ll 1, \quad (8)$$

212 as the scale parameter of the problem. Specifically, $\varepsilon \sim \partial_{x,t}(\star)$ accounts for
213 the slowly-varying downstream modulations of the free surface with respect
214 to space and time.

215 Thus, the governing equations (2) are rewritten in dimensionless terms:

$$\partial_t \rho + \partial_x(\rho u) + \partial_y(\rho v) = 0 \quad (9a)$$

$$\begin{aligned} \rho \varepsilon (\partial_t u + u \partial_x u + v \partial_y u) = & -\varepsilon \partial_x p + \frac{\rho}{Fr} \sin \beta + \\ & + \frac{1}{Re} \left[\partial_{yy} u + \varepsilon^2 \partial_{xx} u + \varepsilon^2 \left(\frac{1}{3} + \vartheta \right) \partial_x (\partial_x u + \partial_y v) \right] \end{aligned} \quad (9b)$$

$$\begin{aligned} \rho \varepsilon^2 (\partial_t v + u \partial_x v + v \partial_y v) = & -\partial_y p - \frac{\rho}{Fr} \cos \beta + \\ & + \frac{\varepsilon}{Re} \left[\varepsilon^2 \partial_{xx} v + \partial_{yy} v + \left(\frac{1}{3} + \vartheta \right) \partial_y (\partial_x u + \partial_y v) \right], \end{aligned} \quad (9c)$$

216 being $Re = \tilde{\rho}_0 \tilde{U}_N \tilde{h}_N / \tilde{\mu}_0$ and $Fr = \tilde{U}_N^2 / g \tilde{h}_N$ the Reynolds number and the
217 Froude number, respectively, with $(x, y, t) \in \mathbb{R} \times [0, h] \times [0, +\infty[$.

218 The system (9) is coupled with the following set of dimensionless bound-

219 ary conditions:

$$u|_0 = v|_0 = 0 \quad (10a)$$

$$Re (1 + \varepsilon^2 \partial_x^2 h) (p|_h - p_i) + \varepsilon \left(\frac{2}{3} - \vartheta \right) (1 + \varepsilon^2 \partial_x^2 h) (\partial_x u|_h + \partial_y v|_h) + \quad (10b)$$

$$-2 \varepsilon (\partial_y v|_h + \varepsilon^2 \partial_x^2 h \partial_x u|_h) + 2 \varepsilon \partial_x h (\partial_y u|_h + \varepsilon^2 \partial_x v|_h) = -\frac{Re}{We} \frac{\varepsilon^2 \partial_{xx} h}{\sqrt{1 + \varepsilon^2 \partial_x^2 h}}$$

$$2 \varepsilon^2 \partial_x h (\partial_y v|_h - \partial_x u|_h) + (1 - \varepsilon^2 \partial_x^2 h) (\partial_y u|_h + \varepsilon^2 \partial_x v|_h) = 0 \quad (10c)$$

$$\partial_t h + u|_h \partial_x h = v|_h, \quad (10d)$$

220 where p_i is the dimensionless atmospheric pressure exerted at the film inter-
 221 face and $We = \tilde{\rho}_0 \tilde{h}_N \tilde{U}_N^2 / \tilde{\gamma}_0$ is the Weber number.

222 3. Low-dimensional modelling

223 Here, the free-surface flow problem is tackled adopting an asymptotic
 224 approximation of the continuity and the Navier-Stokes equations based on
 225 the film aspect ratio $\varepsilon \ll 1$ introduced in § 2.2. A great simplification can
 226 be accomplished by means of a boundary layer approach together with a
 227 depth-averaging technique. Such a procedure leads to the determination of
 228 a reduced coupled system of two equations, having the film thickness $h(x, t)$
 229 and the flow rate per unit of channel width $q(x, t)$ as local dimensionless
 230 unknowns. We propose a two-equation momentum-integral model (MIM)
 231 that is accurate up to and including order $O(\varepsilon^2)$ both in inertial and in
 232 viscous diffusion terms. Based on this approximation, the problem expressed
 233 by (9, 10) will be consistently simplified accounting for the higher magnitude
 234 of surface tension, $We = O(\varepsilon^2)$, compared to inertia-related phenomena,
 235 $Re \sim Fr = O(1)$.

236 Following the classical Polhausen–von Kármán momentum–integral anal-
 237 ysis, the y –momentum equation (9c) and related boundary condition (10b)
 238 serve to eliminate the streamwise pressure gradient term $\partial_x p$ in the x –momentum
 239 equation (9b). Being this term of $O(\varepsilon)$, it is sufficient to retain (9c) and (10b)
 240 up to $O(\varepsilon)$. Differently from the incompressible scenario, in this work, a sup-
 241 plementary constitutive relation is required to describe completely the fluid
 242 system due to the presence of a density term $\rho = \tilde{\rho}/\tilde{\rho}_0 = O(1)$ as an addi-
 243 tional unknown (Richard, 2021).

244 3.1. Barotropic equation of state

245 Since it is difficult to encounter large variations in density in gravity–
 246 driven falling films, we make use of the following linearised Equation of State
 247 (EoS)

$$\tilde{\rho}(\tilde{p}, \tilde{T}, \tilde{S}) = \tilde{\rho}|_{\tilde{h}} + \left(\frac{\partial \tilde{\rho}}{\partial \tilde{p}}\right)_{\tilde{T}, \tilde{S}} (\tilde{p} - \tilde{p}|_{\tilde{h}}) + \left(\frac{\partial \tilde{\rho}}{\partial \tilde{T}}\right)_{\tilde{p}, \tilde{S}} (\tilde{T} - \tilde{T}|_{\tilde{h}}) + \left(\frac{\partial \tilde{\rho}}{\partial \tilde{S}}\right)_{\tilde{p}, \tilde{T}} (\tilde{S} - \tilde{S}|_{\tilde{h}}), \quad (11)$$

248 in the form of a first–order truncated Taylor series expansion as in Batchelor
 249 (2000); Colinet et al. (2001). The validity of (11) is intended to be restricted
 250 to a neighborhood of the reference state, *i.e.* $\tilde{\rho} - \tilde{\rho}|_{\tilde{h}} \ll 1$. Specifically, besides
 251 pressure \tilde{p} , the parameters that characterize such a functional dependence are
 252 the fluid temperature \tilde{T} and its entropy \tilde{S} for a fixed vector of amounts of
 253 constituents.

254 At the present stage, density–affecting thermal effects – which would
 255 have required an energy equation coupling – will be ignored, so as to confine
 256 our current inquiry to a two–equation MIM pattern. Moreover, although the
 257 flow is not itself homentropic, the propagation of small–amplitude long–wave

258 perturbations is shown to be scarcely affected by acoustic attenuation and
 259 dispersion phenomena (Van Dael, 1968; Kinsler et al., 2000). We postpone a
 260 more rigorous proof of this statement to § 6.1, where we deal with the notion
 261 of wave hierarchy.

262 Therefore, the EoS (11) is reduced to a *barotropic* formulation where
 263 density variations with pressure support the propagation of sound waves:

$$\tilde{\rho}(\tilde{p}) = \tilde{\rho}_0 + \left(\frac{\partial \tilde{\rho}}{\partial \tilde{p}} \right)_{\tilde{s}} (\tilde{p} - \tilde{p}|_h). \quad (12)$$

264 Notably, one can refer to the thermodynamic definition of isentropic speed
 265 of sound (Shapiro, 1953)

$$\tilde{a}_0 = \sqrt{\left(\frac{\partial \tilde{p}}{\partial \tilde{\rho}} \right)_{\tilde{s}}}, \quad (13)$$

266 whose magnitude \tilde{a}_0 is supposed to be uniform and constant within $\tilde{\Psi}$, in
 267 order to achieve the dimensionless version of (11), which ultimately reads

$$\rho(p) = 1 + Ma^2 (p - p|_h). \quad (14)$$

268 In (14) an overall Sarrau–Mach number

$$Ma = \frac{\tilde{U}_N}{\tilde{a}_0} \in \mathbb{R}^+, \quad (15)$$

269 expressing the magnitude of inertial forces with respect to elastic ones, has
 270 been introduced as dimensionless group to capture the influence of compress-
 271 ibility on the film flow. As it can be inferred from (14), the classical incom-
 272 pressible limit is recovered as a limiting case when the acoustic propagation
 273 is modelled as an instantaneous phenomenon, *i.e.* $\tilde{a}_0 \rightarrow +\infty \iff Ma \rightarrow 0^+$.

274 *3.1.1. Pressure distribution*

275 Replacement of (14) into the $O(\varepsilon)$ estimate of (9c) leads to the following
 276 first—order linear non—homogeneous Ordinary Differential Equation (ODE)
 277 with respect to the crosswise coordinate y for the film pressure $p(x, y, t)$:

$$\partial_y p + \frac{\cos \beta}{Fr} Ma^2 p = \frac{\cos \beta}{Fr} (Ma^2 p|_h - 1) + \frac{\varepsilon}{Re} \partial_y \mathcal{W} + O(\varepsilon^2), \quad (16)$$

278 in which the function $\mathcal{W}(u, v; \vartheta) = \partial_y v + \left(\frac{1}{3} + \vartheta\right) (\partial_x u + \partial_y v)$ implicitly de-
 279 pends on y through the dimensionless velocity field. The solution of (16),
 280 in which the dimensionless interfacial pressure $p|_h$ has been evaluated using
 281 the normal stress boundary condition (10b), is determined as summation of
 282 the particular solution of (16) and the solution of the corresponding homoge-
 283 neous ODE. The latter is obtained via the method of separation of variables,
 284 whereas the former through the technique of variation of parameters (some-
 285 times referred to as Duhamel’s principle). As a result, the proper solution
 286 of (16) reads:

$$\begin{aligned} p(x, y, t; \vartheta) &= p_i + \frac{\overbrace{\exp\left[\frac{\cos \beta}{Fr} Ma^2 (h - y)\right] - 1}^{\diamond}}{Ma^2} - \frac{\varepsilon^2}{We} \partial_{xx} h + \\ &+ \frac{\varepsilon}{Re} \left(\mathcal{W} - (\partial_x u)|_h \right) + O(\underbrace{\varepsilon Ma^2}_{\clubsuit}), \end{aligned} \quad (17)$$

287 being its full—form given in Appendix A. As expected, as the Mach number
 288 approaches zero, (17) reduces to the pressure distribution obtained by Ruyer-
 289 Quil and Manneville (1998) in the context of a perfectly incompressible free-
 290 surface flow, by virtue of the exponential limit $(e^{m\star} - 1)/\star \rightarrow m$ for vanish-
 291 ing \star (with $m \in \mathbb{R} \setminus \{0\}$), along with the incompressible continuity identity
 292 $\partial_y v = -\partial_x u$.

293 Based on the above considerations, the barotropic EoS (14) can be recast
 294 as

$$\rho(x, y, t; \vartheta) = \underbrace{\exp\left[\frac{\cos\beta}{Fr}Ma^2(h-y)\right]}_{\blacklozenge} + O(\underbrace{\varepsilon Ma^2}_{\clubsuit}). \quad (18)$$

295 *3.1.2. Weak compressibility hypothesis*

296 Although the flow compressibility is taken into account in this model, thin
 297 descending liquid films usually show a weakly compressible behaviour and,
 298 therefore, the expression (18) can be simplified. To do so, the magnitude of
 299 the Mach number can be estimated with respect to ε and, taking inspiration
 300 from Richard (2021), we can write

$$Ma = M \varepsilon^\alpha, \quad (19)$$

301 where α controls the compressibility behaviour and $M = O(1) \in \mathbb{R}_0^+$. As a
 302 consequence, the accuracy of the model is retained only if $\alpha \geq 1$ since the
 303 residual term \clubsuit in (18) is of $O(2\alpha + 1)$. In our model, the Mach number
 304 enters into the governing equations only through the barotropic EoS (14)
 305 and, since $Ma^2 = O(\varepsilon^{2\alpha})$, the different orders in terms of integer power of
 306 the Mach number can be classified as $\alpha = \{1, 3/2, 2, 5/2, \dots\}$.

307 An estimation of the order of magnitude of the exponential term \blacklozenge
 308 in (17, 18) within the low- Ma limit requires one to take the Maclaurin se-
 309 ries expansion $e^\star = \sum_{n=0}^{\infty} (\star^n/n!)$, that, together with the preliminary guess
 310 about the order of magnitude of $Fr = O(1)$, yields to:

$$\underbrace{\exp\left[\frac{\cos\beta}{Fr}Ma^2(h-y)\right]}_{\blacklozenge} \approx 1 + \underbrace{\frac{\cos\beta}{Fr}Ma^2(h-y)}_{\blacklozenge_1} + \frac{1}{2} \underbrace{\left[\frac{\cos\beta}{Fr}Ma^2(h-y)\right]^2}_{\blacklozenge_2}, \quad (20)$$

311 implying that $\blacklozenge_1 = O(\varepsilon^{2\alpha})$ and $\blacklozenge_2 = O(\varepsilon^{4\alpha})$. Depending on the value of α ,
 312 a twofold level of compressibility can be consequently addressed in view of
 313 the prescribed $O(\varepsilon^2)$ accuracy criterion:

$$\rho(x, y, t; \vartheta) = \begin{cases} 1 + O(\varepsilon^3), & \alpha \geq \frac{3}{2} \\ 1 + \frac{\cos\beta}{Fr} Ma^2 (h - y) + O(\varepsilon^3), & \alpha = 1. \end{cases} \quad (21)$$

314 Thus, when $\alpha \geq 3/2$ the analysis is formally identical to the incompressible
 315 scenario, since a relation of asymptotic equivalence holds between $\tilde{\rho}(x, y, t)$
 316 and $\tilde{\rho}_0$. In other words, the relation (19) provides a rule-of-thumb crite-
 317 rion for the film flow to be considered as weakly-compressible in asymptotic
 318 terms. For example, if we assume $\varepsilon = 0.01$ as long-wave parameter (jointly
 319 with a unitary-valued M), we find the threshold for incompressibility as
 320 $Ma \lesssim 0.001$.

321 3.2. Boundary layer equations

322 In this this paper we focus on the weakly-compressible regime correspond-
 323 ing to $\alpha = 1$. In this scenario, the derivative of the pressure distribution (17)
 324 is computed using the expression (21) with $\alpha = 1$, leading to

$$\begin{aligned} \partial_x p(x, y, t; \vartheta) = & \frac{\cos\beta}{Fr} \partial_x h - \frac{\varepsilon^2}{We} \partial_{xxx} h + \\ & + \frac{\varepsilon}{Re} \left[\partial_{xy} v + \left(\frac{1}{3} + \vartheta \right) \partial_x (\partial_x u + \partial_y v) - \partial_x ((\partial_x u)|_h) \right] + O(\varepsilon^2). \end{aligned} \quad (22)$$

325 As mentioned above, (22) is now substituted in lieu of $\partial_x p$ in (9b), showing
 326 that ϑ -dependent contributions mutually cancel themselves out.

327 Then, the replacement of ρ and $\partial_x p$ jointly permits obtaining the second-
 328 order set of weakly compressible Boundary Layer Equations (BLEs), which

329 finally reads:

$$\partial_x u + \partial_y v + \frac{\cos \beta}{Fr} Ma^2 (\partial_t h + u \partial_x h - v) = 0 \quad (23a)$$

$$\begin{aligned} \varepsilon (\partial_t u + u \partial_x u + v \partial_y u) &= \frac{\partial_{yy} u}{Re} + \frac{\varepsilon^3}{We} \partial_{xxx} h - \varepsilon \frac{\cos \beta}{Fr} \partial_x h + \\ &+ \frac{\sin \beta}{Fr} \left(1 + \frac{\cos \beta}{Fr} Ma^2 (h - y) \right) + \frac{\varepsilon^2}{Re} \left[\partial_{xx} u - \partial_{xy} v + \partial_x ((\partial_x u)|_h) \right]. \end{aligned} \quad (23b)$$

330 By resorting to Leibniz's integral rule, BLEs (23) are integrated over the
 331 depth $\int_0^h (\star) dy$ to reduce the space dimensionality of the problem. The basic
 332 idea behind this modelling strategy is the elimination of the cross-stream flow
 333 dependency (Ruyer-Quil and Manneville, 2000).

334 Unfortunately, the resulting BLEs fail to be entirely expressed in terms of
 335 the local film thickness $h(x, t)$ and the local flow rate $q(x, t) = \int_0^h u(y) dy$.
 336 Thus, closure laws are needed in (23b) for the following terms: the so-
 337 called shape factor $\int_0^h u^2 dy$, the difference between interfacial and wall shear
 338 stresses $((\partial_y u)|_h - (\partial_y u)|_0)$, and the antiderivative of other second-order terms
 339 within square brackets ($\propto \varepsilon^2/Re$). Moreover, since the compressibility intro-
 340 duces a novel second-order contribution, related to the crosswise component
 341 of velocity, *viz.* $\int_0^h v dy$, in (23a) an additional closure is required. Such
 342 closures can be obtained via the explicit expression for the unknown velocity
 343 field $u(x, y, t)$, $v(x, y, t)$.

344 3.2.1. Long-wave approximation

345 In this work, we adopt a long wave approach following the classical Ben-
 346 ney's closure technique (Benney, 1966; Gjevik, 1970; Lin, 1974; Chang, 1986).
 347 Accordingly, each variable $\mathcal{V} = \{u, v, p, \rho\}$ appearing in the primitive prob-
 348 lem is decomposed as a formal power-series regular perturbation expansion,

349 having ε as basis:

$$\mathcal{V}^{(\varepsilon)} = \mathcal{V}^{(0)} + \varepsilon \mathcal{V}^{(1)} + \varepsilon^2 \mathcal{V}^{(2)} + \dots \quad (24)$$

350 The right-hand side of (24) is ideal for assessing the effect of a small per-
351 turbation in ε about zero, provided that proper accuracy constraints are
352 met (Simmonds and Mann Jr, 1998). Specifically, mathematical convergence
353 of the infinite series (24) is not necessary (Jeffreys, 1926; Van Dyke and
354 Rosenblat, 1975). On the other hand, it is required that – once truncated –
355 $\mathcal{V}^{(\varepsilon)}$ rapidly approaches \mathcal{V} in the limit of vanishing ε . This is equivalent to
356 enforce that the approximation error $|\mathcal{V} - \mathcal{V}^{(\varepsilon)}|$ scales as the first neglected
357 term of the series (24). By assuming this residue to be $\sim \varepsilon^3$, the $O(\varepsilon^2)$
358 truncation of the previous ansatz (24) can be then substituted in (9, 10, 14),
359 allowing the corresponding equations to be broken up into different orders
360 and sequentially solved. Specifically, the $O(\varepsilon^0, \varepsilon^1)$ restrictions of the prob-
361 lem coincide with their respective incompressible versions, due to the fact
362 that Ma -related influence intervenes only at $O(\varepsilon^2)$ when $\alpha = 1$, through the
363 equality $\rho^{(2)} = M^2 (p^{(0)} - p^{(0)}|_h)$ by (14). Also, the terms including the ex-
364 pansion viscosity appear to be irrelevant, due to the fact that the $O(\varepsilon^0, \varepsilon^1)$
365 velocity fields are solenoidal. Hence, a comparison between the compressible
366 second-order profiles and their incompressible analogues will be helpful to
367 understand the impact of a varying density on flow-related quantities; this
368 aspect will be discussed in § 6.2.

369 3.3. Depth-averaged model

370 Upon substitution, we now take advantage of the expressions for the
371 asymptotic expansions determined beforehand. These are confined to Ap-

372 pendix B only for the sake of brevity.

373 In order to derive the depth–integral model, three steps need to be per-
 374 formed: (i) replace higher–order time derivatives of h by virtue of a consis-
 375 tent estimate of the kinematic boundary condition (10d), (ii) replace space
 376 derivatives of q – except for the diffusive term $\partial_{xx}q$ – by the corresponding
 377 consistent asymptotic expansions, and (iii) add to the r.h.s. of (23b) the
 378 higher–order residue $+3(q^{(0)} + \varepsilon q^{(1)} + \varepsilon^2 q^{(2)} - q) / Re h^2 = O(\varepsilon^3)$, so to pre-
 379 clude algebraic cancellation of linear source terms – see (26) – as part of the
 380 model quasi–linear reformulation (Lavalle et al., 2015). After these manip-
 381 ulations, the following depth–averaged closed set of two evolution equations
 382 is obtained:

$$\partial_t h + \partial_x q - \frac{\Lambda \cos \beta h^3 (\partial_x h)}{2 Fr} Ma^2 = 0 \quad (25a)$$

$$\begin{aligned} & \frac{h (\partial_x h) \cos \beta \varepsilon}{Fr} + \frac{3 h^4 (\partial_x h) \Lambda^2 \varepsilon}{5} + \varepsilon (\partial_t q) = \frac{h (\partial_{xxx} h) \varepsilon^3}{We} + \quad (25b) \\ & - \frac{4 Re h^5 (\partial_{xxxx} h) \Lambda \varepsilon^4}{21 We} - \frac{2 Re h^4 (\partial_x h) (\partial_{xxx} h) \Lambda \varepsilon^4}{3 We} + \\ & + \frac{2 Re h^4 (\partial_{xx} h)^2 \Lambda \varepsilon^4}{5 We} + \frac{4 Re h^3 (\partial_x h)^2 (\partial_{xx} h) \Lambda \varepsilon^4}{5 We} + \\ & + \frac{4 Re h^5 (\partial_{xx} h) \Lambda \cos \beta \varepsilon^2}{21 Fr} + \frac{16 Re h^4 (\partial_x h)^2 \Lambda \cos \beta \varepsilon^2}{15 Fr} + \\ & + \frac{3 Ma^2 h^2 \Lambda \cos \beta}{8 Fr Re} - \frac{8 Re h^8 (\partial_{xx} h) \Lambda^3 \varepsilon^2}{105} - \frac{23 Re h^7 (\partial_x h)^2 \Lambda^3 \varepsilon^2}{35} + \\ & + \frac{h^2 (\partial_{xx} h) \Lambda \varepsilon^2}{Re} + \frac{3 h (\partial_x h)^2 \Lambda \varepsilon^2}{Re} + \frac{2 (\partial_{xx} q) \varepsilon^2}{Re} + \frac{h \Lambda}{Re} - \frac{3 q}{Re h^2}, \end{aligned}$$

383 where the dimensionless number Λ is defined as $Re/Fr \sin \beta$. Here, by using
 384 the definitions of \tilde{U}_N , Re and Fr , we get that $\Lambda = 3$. In other contexts, this
 385 parameter may assume different values, such as when a different character-
 386 istic speed is used instead of Nusselt integral velocity \tilde{U}_N , in case of a fluid

387 exhibiting a non-Newtonian constitutive behaviour (Noble and Vila, 2013),
388 or in presence of a variable or uneven interfacial pressure p_i ; it has been de-
389 cided not to replace Λ by any numerical value (Richard et al., 2019) only to
390 prevent loss of generality.

391 With reference to equation (25b), it is worth pointing out two additional
392 facts. (i) Higher-order and non-linear capillary terms have been explicitly
393 and fully retained, unlike what customarily developed (Ruyer-Quil and Man-
394 neville, 1998; Richard et al., 2016, 2019). In fact, their contribution could be
395 equally gathered on the l.h.s. within the canonical convective term propor-
396 tional to $\varepsilon \partial_x (q^2/h)$, leading to an equivalent model in terms of consistency.
397 (ii) Inertial terms have been maintained up to $O(\varepsilon^2)$, dissimilarly from the
398 well-established practice of relying on a simplified model (Ruyer-Quil and
399 Manneville, 2002). In fact, we are interested in comparing the whole second-
400 order expansions with their incompressible analogues.

401 The derived shallow-water system (25) constitutes a second-order re-
402 duced model describing the weakly-compressible free-surface flow of a wavy
403 gravity-driven Newtonian falling film. In the scenario where the temperature
404 field within the liquid film yields density variations, the EoS (12) should be
405 modified accordingly to take into account density-affecting thermal effects.
406 In addition, the model (25) should be coupled to an integral form of the
407 energy equation to characterise the interplay between hydrodynamics, com-
408 pressibility and heat transfer. For this, reduced models for non-isothermal
409 (incompressible) falling films have been successful in solving the heat transfer
410 across the liquid film (Trevelyan et al., 2007; Thompson et al., 2019; Cellier
411 and Ruyer-Quil, 2020).

412 **4. Temporal linear stability**

413 A temporal stability analysis relies on the existence of a steady solution
 414 about which perturbations are superimposed. Let $\mathbf{Q}(x, t) = \{h(x, t), q(x, t)\}^T$
 415 represent the column vector containing the two unknown integral variables
 416 describing the film descent. Indeed, the weakly-compressible shallow-water
 417 equations (25) possibly admit to be recast as

$$\partial_t \mathbf{Q} + \partial_x \mathcal{F}(\mathbf{Q}) = \mathcal{S}(\mathbf{Q}), \quad (26)$$

418 where \mathcal{F} is the associated flux vector whereas \mathcal{S} gathers source terms to-
 419 gether (Noble and Vila, 2014).

420 *4.1. Normal mode analysis*

421 The linear stability problem of the low-dimensional weakly-compressible
 422 model (25) is approached through normal mode decomposition, according to
 423 which a harmonic infinitesimal disturbance \mathbf{Q}_p , having $\|\hat{\mathbf{Q}}\| \ll 1$ as ampli-
 424 tude, is added to the Nusselt base state. The latter is explained in terms
 425 of the dimensionless uniform parallel solution $\mathbf{Q}_0 = \{h_0, q_0\}^T$, in which
 426 $h_0 = \tilde{h}_0/\tilde{h}_N \equiv 1$ by definition, whereas the novel expression for the com-
 427 pressible primary discharge q_0 will be disclosed as part of the linearisation
 428 process. Accordingly, it is written

$$\mathbf{Q}(x, t) = \mathbf{Q}_0 + \mathbf{Q}_p(x, t) \quad (27a)$$

$$\mathbf{Q}_p(x, t) = \hat{\mathbf{Q}} \exp[i k (x - c t)], \quad (27b)$$

429 where it remains understood that $k = 2\pi \tilde{h}_N/\tilde{\mathcal{L}} \in \mathbb{R}^+$ and $c = c_r + i c_i \in \mathbb{C}$
 430 are, respectively, the dimensionless real wave-number and the complex wave

431 celerity of the propagating sine-type pulse. In particular, c_r accounts for its
 432 phase velocity, whereas $k c_i$ determines its degree of amplification or damping,
 433 depending on its sign: with reference to (27b), instability of the mean flow
 434 evidently sets in on the condition that $k c_i > 0$.

435 4.1.1. Base flow calculation

436 Quasi-linear conservation form (26) actually stipulates a formal relation
 437 between differential operators (Meliga et al., 2010) in such a way that

$$\mathcal{S}(\mathbf{Q}_0) = 0 \quad (28)$$

438 restores the equilibrium condition constraining the dimensionless compress-
 439 ible base flow rate $q_0(Re, \beta, Ma)$ to the dimensionless waveless thickness h_0 .
 440 Solving (28) we find:

$$q_0 = \frac{\Lambda h_0^3}{3} \left(1 + \overbrace{\frac{3}{8} \frac{Ma^2 \Lambda \cot \beta h_0}{Re}}^{\Delta q_{0, \text{rel}}^{(2)}} \right), \quad (29)$$

441 which explicitly shows that compressibility entails a relative increase in the
 442 equilibrium flow rate q_0 , according to the over-bracketed second-order contri-
 443 bution denoted as $\Delta q_{0, \text{rel}}^{(2)}$, with respect to its incompressible limit
 444 $q_0^{Ma \rightarrow 0^+} = \Lambda h_0^3/3$. Expression (29) likewise coincides with the stationary
 445 waveless solution associated to system (23) in the case of unidirectional flow.
 446 Compressible effects are kept at the base flow level \mathbf{Q}_0 , on which linear distur-
 447 bances \mathbf{Q}_p develop, by means of a small additive contribution to the incom-
 448 pressible ground-state flow rate $q_0^{Ma \rightarrow 0^+}$. Such a correction ($q_0^{Ma \rightarrow 0^+} \Delta q_{0, \text{rel}}^{(2)}$)
 449 appears to be of $O(\varepsilon^2)$ since, choosing $\alpha = 1$, we assumed Ma to be of order
 450 $O(\varepsilon)$.

451 *4.1.2. Model dispersion relation*

452 Dropping higher-order perturbations and plugging (27) into (25) yields
 453 the following matrix-form differential system:

$$\begin{aligned} \partial_t \mathbf{Q}_p + \begin{bmatrix} a_{11} & 1 \\ a_{21} & 0 \end{bmatrix} \partial_x \mathbf{Q}_p = \begin{bmatrix} 0 & 0 \\ b_{21} & b_{22} \end{bmatrix} \mathbf{Q}_p + \\ + \begin{bmatrix} 0 & 0 \\ c_{21} & c_{22} \end{bmatrix} \partial_{xx} \mathbf{Q}_p + \begin{bmatrix} 0 & 0 \\ s_{21} & 0 \end{bmatrix} \partial_{xxx} \mathbf{Q}_p + \begin{bmatrix} 0 & 0 \\ d_{21} & 0 \end{bmatrix} \partial_{xxxx} \mathbf{Q}_p, \end{aligned} \quad (30)$$

454 where

$$a_{11} = -\frac{Ma^2 h_0^3 \Lambda^2 \cot \beta}{2 Re} \quad b_{21} = \frac{3 \Lambda}{Re} + \frac{3 Ma^2 h_0 \Lambda^2 \cot \beta}{2 Re^2} \quad (31a)$$

$$a_{21} = \frac{3}{5} h_0^4 \Lambda^2 + \frac{h_0 \Lambda \cot \beta}{Re} \quad b_{22} = -\frac{3}{Re h_0^2} \quad c_{22} = \frac{2}{Re} \quad s_{21} = \frac{h_0}{We} \quad (31b)$$

$$c_{21} = \frac{4}{21} h_0^5 \Lambda^2 \cot \beta - \frac{8}{105} Re h_0^8 \Lambda^3 + \frac{h_0^2 \Lambda}{Re} \quad d_{21} = -\frac{4}{21} \frac{Re h_0^5 \Lambda}{We}. \quad (31c)$$

455 Expressions (31b – 31c) do not incorporate the Mach number, thus ε has
 456 been legitimately replaced there by a unitary value (Richard et al., 2019).
 457 Such assignment is based on the fact that pertinent orders of magnitude have
 458 been already accounted for in the integral model (25).

459 Equation (30) accounts for the normal mode evolution (27b) under the
 460 form of a generalised algebraic eigenvalue problem for c and $\hat{\mathbf{Q}}$, having
 461 $\langle k; Re, \beta, We, Ma \rangle$ as independent set of relevant parameters. Seeking
 462 a non-trivial solution, one has to impose that the matrix associated to the
 463 linearised system is degenerate. This leads to a quadratic polynomial dis-
 464 persion relation over the complex field in the phase speed c with complex
 465 k -dependent coefficients, written as

$$\begin{aligned}
& -k c^2 + [a_{11} k + i (b_{22} - k^2 c_{22})] c + \\
& + k^3 s_{21} + k a_{21} + i [d_{21} k^4 + (a_{11} c_{22} - c_{21}) k^2 + (b_{21} - a_{11} b_{22})] = 0. \quad (32)
\end{aligned}$$

466 *4.2. Celerity long-wave expansion*

467 Following Yih (1963), we consider the temporal stability problem in terms
468 of an asymptotic expansion of the wave celerity $c(k)$ into successive powers
469 of the wavenumber k :

$$c = c^{(0)} + k c^{(1)} + k^2 c^{(2)} + k^3 c^{(3)} + \dots, \quad (33)$$

470 within the limit provided by the long-wave approximation ($k \ll 1$) assumed
471 in this work. In analogy with the closure algorithm illustrated in § 3.2.1,
472 the expansion (33) is substituted into the dispersion relation (32). Ensuring
473 that each order in k satisfies (32), we get a cascade of equations from which
474 the higher-order celerities $c^{(n)}(k)$ ($n = 0, 1, 2, \dots$) are obtained. Although
475 evolution equations (25) are consistent up to $O(\varepsilon^2)$, we intentionally take
476 the expansion (33) for the celerity $c(k)$ up to its successive order in terms of
477 k , that is until $O(k^3)$. In this way, we can test the accuracy of the present
478 model (25) in its incompressible limit $Ma \rightarrow 0^+$, by setting the benchmark
479 against the Orr–Sommerfeld stability problem (Orr, 1907; Sommerfeld, 1908)
480 at the corresponding order. Such a comparative approach constitutes a well-
481 trodden path among the falling-film community (Ruyer-Quil and Manneville,
482 1998; Samanta et al., 2011; Samanta, 2014; Richard et al., 2016). Specifically,

483 we obtain:

$$c^{(0)} = 3 \quad (34a)$$

$$c^{(1)} = 3i \left(\frac{2}{5} Re - \frac{1}{3} \cot \beta + \Gamma_2^{(1)} Ma^2 \cot \beta \right) \quad (34b)$$

$$c^{(2)} = 3 \left(-1 + \frac{10}{21} Re \cot \beta - \frac{4}{7} Re^2 + \Gamma_2^{(2)} Ma^2 \cot \beta + \Gamma_4^{(2)} Ma^4 \cot^2 \beta \right) \quad (34c)$$

$$c^{(3)} = 3i \left(-\frac{1}{9} Re \cot^2 \beta + \frac{128}{105} Re^2 \cot \beta + \frac{2}{9} \cot \beta - \frac{Re}{9We} - \frac{228}{175} Re^3 + \right. \quad (34d)$$

$$\left. -\frac{34}{15} Re + \Gamma_2^{(3)} Ma^2 \cot \beta + \Gamma_4^{(3)} Ma^4 \cot^2 \beta + \Gamma_6^{(3)} Ma^6 \cot^3 \beta \right),$$

484 in which we use the equality $\Lambda = 3$ and the identity $h_0 \equiv 1$. Those expres-
 485 sions for the wave celerities have been written to highlight the effect of the
 486 compressibility. In fact, the expansions (34) are impacted by compressibility
 487 from $n = 1$ onwards ($n = 1, 2, \dots$) through additive contributions that take
 488 the form $\Gamma_{2j}^{(n)} Ma^{2j} \cot^j \beta$, with $1 \leq j \leq n$. These are found to be:

$$\Gamma_2^{(1)} = \frac{3}{2} \quad \Gamma_2^{(2)} = \frac{1}{2} \cot \beta - \frac{18}{5} Re - \frac{1}{Re} \quad (35a)$$

$$\Gamma_4^{(2)} = -\frac{9}{4} \quad \Gamma_2^{(3)} = \frac{19}{7} Re \cot \beta - \frac{324}{35} Re^2 - \frac{9}{2} \quad (35b)$$

$$\Gamma_4^{(3)} = \frac{3}{4} \cot \beta - \frac{243}{20} Re - \frac{3}{2Re} \quad \Gamma_6^{(3)} = -\frac{27}{8}. \quad (35c)$$

489 In accordance with the adopted standard of accuracy, the current model
 490 is consistent with the asymptotic expansions of solutions to Orr–Sommerfeld
 491 boundary–value problem, reported in Ruyer-Quil and Manneville (1998), in
 492 the limit of $Ma \rightarrow 0^+$: (25) is able to correctly recover $c^{(0)}$, $c^{(1)}|_{Ma \rightarrow 0^+}$ and
 493 $c^{(2)}|_{Ma \rightarrow 0^+}$, but it manifests disagreements on successive orders. For more
 494 in–depth reflection on such validation the reader is referred to Appendix
 495 C, being the primary focus of sections §§ 4, 5 upon the influence of a weak
 496 compressibility on the linear stability.

497 5. Results and discussion

498 In this section we examine the relations (34) in the light of the well-known
499 results from Kapitza (1948) and Benjamin (1957). The $O(k^0)$ celerity (34a)
500 immediately captures the classical phase speed of free-surface waves, which
501 travel three times faster than the averaged flat film, regardless of its com-
502 pressible behavior. Due to the nature of (14) as EoS, the compressibility
503 terms controlled by the Mach number affect only even powers Ma^{2j} through-
504 out $O(k^n)$ expansions (34b – 34d), for $1 \leq j \leq n$.

505 Secondly, as evidenced by the relations (34) and (31a), a vertical liquid
506 film is not affected by the compressibility since $\cot\left(\frac{\pi}{2}\right) = 0$ in (25). On
507 the other hand, when the plate is horizontal, $\beta = 0$ and no gravity-driven
508 drainage is possible.

509 5.1. Impact of compressibility on the wave celerity

510 Differently from the incompressible Navier–Stokes equations, whose tem-
511 poral stability analysis is pursued through numerical solution of the Orr–
512 Sommerfeld fourth-order differential problem in the cross-stream coordinate,
513 in this case the dispersion relation (32) is a quadratic polynomial equation
514 in $c(k)$, which is easily solvable numerically.

515 Initially, we consider a falling liquid film whose incompressible flow is
516 marginally stable. This case will be shown to be the most favourable to
517 discern compressibility-related effects on the film flow stability within the
518 investigated weakly-compressible regime. The plate is angled at $\beta = 4.6^\circ$.
519 As an aside, this choice enables us to compare the wave celerity and growth
520 rate (see Appendix C) between the results presented here within the incom-

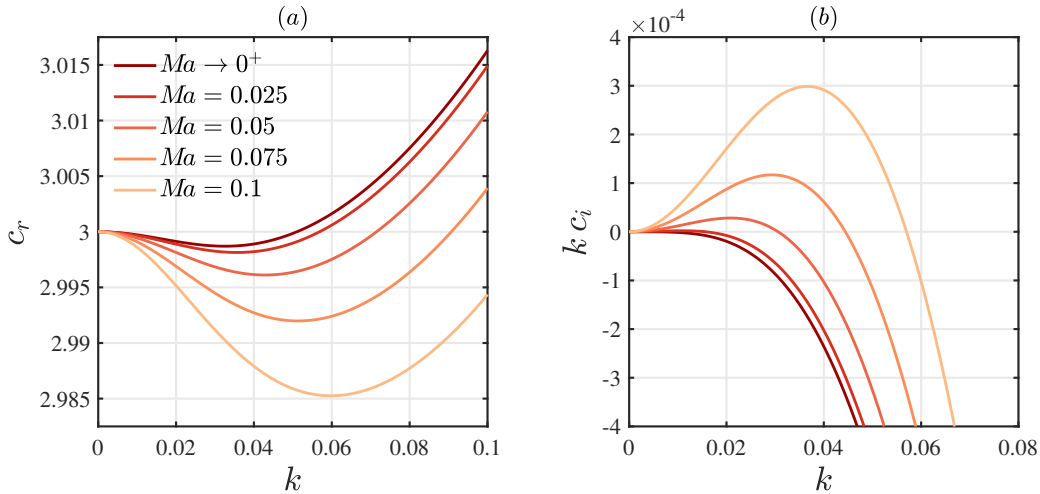


Figure 2: Impact of compressibility on the graphical representation of solutions to the dispersion relation (32) for the second-order integral model (25), in terms of (a) phase speed c_r and (b) growth rate $k c_i$ as a function of the dimensionless wavenumber k , for different small values of the Mach number Ma , displayed in the legend. The axes are dimensionless. The data used are taken from Brevdo et al. (1999) and correspond to the following set of values: $g = 9.81 \text{ m s}^{-2}$, $\beta = 4.6^\circ$, $Re = 5/6 \cot \beta = 10.357$, $\tilde{\rho}_0 = 1130 \text{ kg m}^{-3}$, $\tilde{\mu}_0 = 5.673 \cdot 10^{-3} \text{ Pa s}$, $\tilde{\gamma}_0 = 69.0 \cdot 10^{-3} \text{ N m}^{-1}$. Comparison with Brevdo et al. (1999) is shown in Appendix C for the incompressible scenario.

521 pressible limit $Ma \rightarrow 0^+$ (dark red line in figure 2) and those determined
522 by Brevdo et al. (1999) for a perfectly incompressible falling film in a passive
523 atmosphere. The effects of compressibility on the hydraulic branch solv-
524 ing (32) both in its real and imaginary parts are displayed in figure 2a, b
525 respectively, for sufficiently small values of the Mach number $Ma = O(\varepsilon)$.
526 Specifically, the evolution of the phase speed $c_r(k)$ bends downwards as the
527 Mach number increases. Nonetheless, the same long-wave limit $c^{(0)}$ is re-
528 covered, as established by (34a). The delaying effect of compressibility on
529 the phase velocity of linear waves (Richard et al., 2019) finds confirmation in

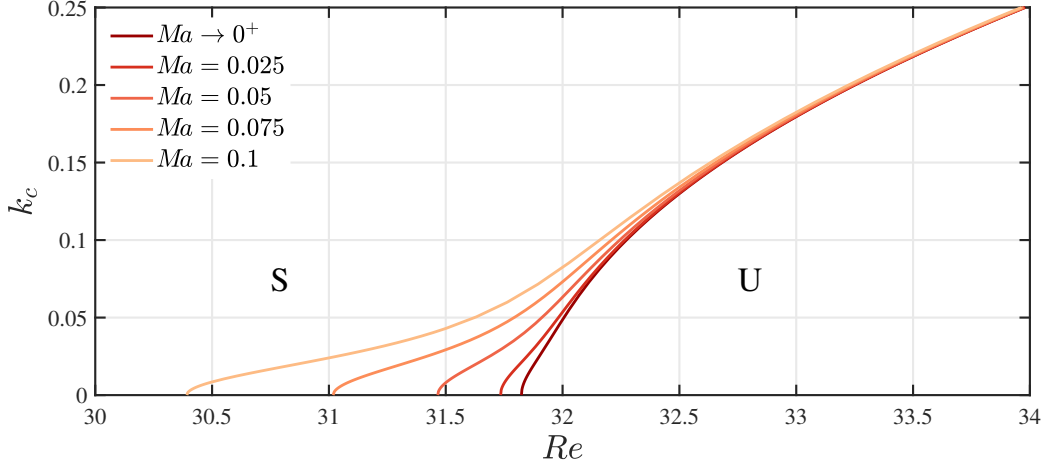


Figure 3: Impact of compressibility on the neutral stability diagram displaying the dimensionless cut-off wavenumber k_c as a function of the Reynolds number Re , for different small values of the Mach number Ma , shown in the legend. Parameter values: $g = 9.81 \text{ m s}^{-2}$, $\beta = 1.5^\circ$. Fluid physical properties – related to a falling film consisting of a water–glycerin mixture – are taken from Liu and Gollub (1994): $\tilde{\rho}_0 = 1070 \text{ kg m}^{-3}$, $\tilde{\mu}_0 = 6.72 \cdot 10^{-3} \text{ Pa s}$, $\tilde{\gamma}_0 = 67.0 \cdot 10^{-3} \text{ N m}^{-1}$. The *stable* and *unstable* domain in the (Ma, Re) plane corresponds to areas labelled, respectively, “S” and “U”.

530 our study. The growth rate $k c_i(k)$ shown in figure 2b deviates upwards and
 531 towards increasing cut-off wavenumber k_c for growing Ma . Thus, compress-
 532 ibility plays a destabilising role on linear free-surface waves.

533 Even more distinctly, we observe this feature in figure 3, which shows the
 534 curve of marginal stability obtained for different values of the Mach number
 535 in the (Re, k_c) plane for $\beta = 1.5^\circ$. Above the marginal stability curve, per-
 536 turbations of wavenumber k decay in time, whereas they are amplified below.
 537 Here, the unstable region systematically undergoes a non-linear enlargement
 538 up to a smaller critical Reynolds number Re_{cr} due to the compressibility.

539 In order to quantify this shift into the stability threshold, we can examine

540 the first-order expansion of the wave celerity $c^{(1)}$, which for $Ma \ll 1$ yields
 541 the following relation

$$Re_{cr} = \frac{5}{6} \left(1 - \frac{9}{2} Ma^2 \right) \cot \beta, \quad (36)$$

542 obtained by making Re explicit from (34b) when the neutral stability condi-
 543 tion $k c_i(k_c) = 0 \iff c^{(1)}|_{Re_{cr}} = 0$ is imposed. In the limit of null Mach num-
 544 ber, equation (36) reduces to the result of Benjamin (1957) and Yih (1963),
 545 *i.e.*, $Re_{cr}^{Ma \rightarrow 0^+} = 5/6 \cot \beta$. Conversely, we observe that for $Ma = O(\varepsilon) > 0$
 546 the compressibility lowers the critical Reynolds number Re_{cr} by a factor equal
 547 to

$$\frac{Re_{cr}(Ma)}{Re_{cr}^{Ma \rightarrow 0^+}} = 1 - \frac{9}{2} Ma^2 < 1, \quad (37)$$

548 anticipating the flow primary instability. This effect tends to asymptotically
 549 vanish in highly inertial regimes, within which compressible curves visibly be-
 550 come rapidly convergent towards the incompressible marginal stability plot
 551 (right-most line in figure 3). This finding is consistent as both the two com-
 552 pressible coefficients (31a) of the eigen-problem (30) are inversely propor-
 553 tional to the Reynolds number or its square power. Interestingly, we remark
 554 that the ratio expressed by (37) is independent of the plate inclination β .

555 5.2. Parametric analysis

556 Aiming at understanding the basic effects of compressibility on the film
 557 destabilisation, we investigate how the growth rate of disturbances $k c_i$ evolves
 558 as the parameter space, namely $\langle Re, \beta, We, Ma \rangle$, is explored. This will en-
 559 able us to understand the fundamental physical mechanism through which
 560 compressibility acts, which we examine more in depth in § 6.

561 We start by providing a variety of numerical solutions to the linear sta-
562 bility problem (30) within the plane $(k, k c_i)$, for different values of the
563 Reynolds number Re and angle of inclination β . Equations (34) suggest
564 that a polynomial-type dependence is established by the novel Ma^{2j} -related
565 contributions, namely $\Gamma_{2j}^{(n)} \cot^j \beta$. Unfortunately, the coefficients $\Gamma_{2j}^{(n)}$ display
566 a fairly cumbersome functional dependence on $\cot \beta$ (as well as on Re) –
567 apart from when $j = n$. For this reason, notwithstanding that the compress-
568 ibility has no impact on a vertical falling film, it is not possible to determine
569 *a priori* whether its effects varies with the inclination. Therefore, we will
570 extensively cover the full range of variability in β , starting by focusing on
571 mildly tilted configurations.

572 In figure 4 we initially consider four cases, denoted with letters $(a-d)$,
573 which differ from each other in terms of slope. To draw an appropriate
574 comparison among these scenarios between each compressible curve ($Ma =$
575 0.1 – dashed lines) and its incompressible counterpart (solid lines), the so-
576 defined Reynolds critical ratio RCR

$$\text{RCR} \stackrel{\text{def}}{=} \frac{Re}{Re_{cr}^{Ma \rightarrow 0^+}} \quad (38)$$

577 is introduced as an inertia-based parameter. Four growing values of RCR
578 are considered in each of the panels of figure 4, starting from a value which
579 is numerically less than unity – which indicates a stable situation for a per-
580 fectly incompressible falling film flow – before moving to values of Re which
581 progressively exceed the critical incompressible threshold.

582 As expected, the augmentation of RCR is associated with the extension
583 of the instability region $c_i(k) > 0$. When we switch from each incompressible
584 plot to its compressible analogue, the rightwards shift of the cut-off wavenum-

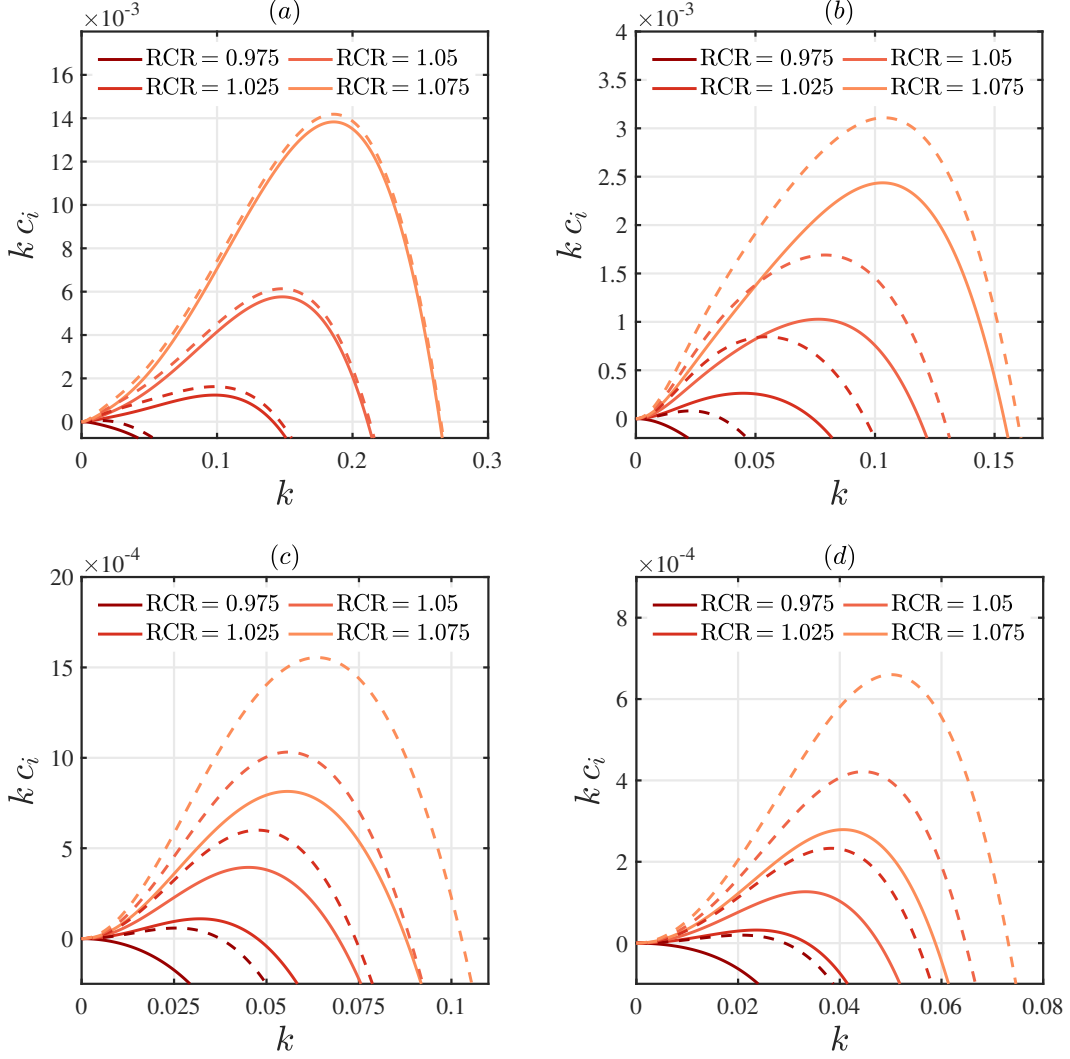


Figure 4: Effect of the Reynolds critical ratio RCR (38) (shown in the legend) on the graphical representation of the solution to the dispersion relation (32) for the derived weakly-compressible second-order model (25), in terms of the dimensionless imaginary growth rate $k c_i$ as a function of the dimensionless wavenumber k , for flow configurations which differ from each other in the value of the inclination angle β : (a) $\beta = 1.5^\circ$, (b) $\beta = 3.0^\circ$, (c) $\beta = 6.0^\circ$, (d) $\beta = 12.0^\circ$. The axes are dimensionless. Solid lines: $Ma \rightarrow 0^+$ (incompressible case), dashed lines: $Ma = 0.1$. Apart from the tilt angle β , other parameter values and fluid physical properties employed here are those of figure 3.

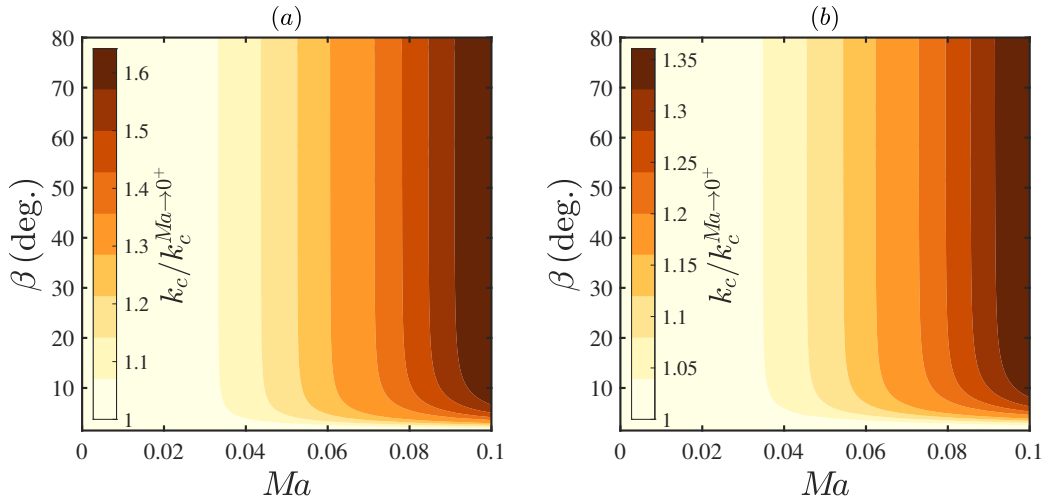


Figure 5: Effect of the Mach number $Ma = O(\varepsilon)$ and of the angle of inclination β on the stability of a falling water–glycerin film in terms of deviation of the cut–off wavenumber k_c from its incompressible limit $k_c^{Ma \rightarrow 0^+}$ with reference to the temporal growth rate of linear disturbances $k c_i(k)$, for two different fixed values of the Reynolds critical ratio (38), corresponding to (a) $\text{RCR} = 1.025$ and (b) $\text{RCR} = 1.05$. In overall terms, darker regions correspond to a greater destabilisation. The set of parameter values and fluid physical properties is the same specified for figure 3.

585 ber k_c is reduced as the RCR is raised. This is in accordance with what
 586 previously shown in figure 3. As the incline of the plate becomes steeper,
 587 provided that moderately low–angle configurations are explored, the com-
 588 pressibility plays an increasingly important effect in relative terms in terms
 589 of a rightward shift of the dispersion curves.

590 In order to better appreciate this phenomenon, we represent in figure 5
 591 the contours of the cut–off wavenumber related to its incompressible limit
 592 $k_c/k_c^{Ma \rightarrow 0^+}$ as a function of the Mach number Ma and of the inclination
 593 angle β for two different values of Reynolds critical ratio RCR beyond the

594 stability threshold, corresponding to (a) $\text{RCR} = 1.025$ and (b) $\text{RCR} = 1.05$,
595 respectively. In both scenarios we identify two distinct regions of the (Ma, β)
596 plane: (i) a low-angle region ($1.5^\circ \lesssim \beta \lesssim 12^\circ$) where compressibility-induced
597 destabilisation is not fully-developed in terms of rightward shift of the cut-
598 off wavenumber and (ii) a region that covers moderately to highly tilted
599 configurations ($12^\circ \lesssim \beta \lesssim 80^\circ$), where the same effects are independent of
600 the value of inclination angle β . From a graphical point of view, the isolines
601 rapidly tend to become vertical, indicating a fast saturation of $k_c/k_c^{Ma \rightarrow 0^+}$
602 with respect to slope.

603 Within area (ii), at $Ma = 0.1$ – the highest level of weak compressibility
604 investigated – the cut-off wavenumber is increased by up to roughly 60%
605 when $\text{RCR} = 1.025$ and 35% when $\text{RCR} = 1.05$ in comparison with the in-
606 compressible case. As it will soon become clear, there exists a third upper
607 region (iii) – for $80^\circ \lesssim \beta \leq 90^\circ$ – which is difficult to explore by employ-
608 ing the parameter RCR since, there, a vertically falling film flow is always
609 unstable to linear perturbations (Benjamin, 1957; Yih, 1963).

610 A similar behavior is shown by the most unstable wavenumber and the
611 maximum growth rate of linear disturbances related to their incompressible
612 limit, *viz.* $k_{\max}/k_{\max}^{Ma \rightarrow 0^+}$ and $\omega_{i, \max}/\omega_{i, \max}^{Ma \rightarrow 0^+}$ respectively, which are displayed
613 in figure 6*a, b* as a function of the Mach number Ma and the inclination
614 angle β in the case of a Reynolds critical ratio equal to $\text{RCR} = 1.05$. The
615 results are shown up to $\beta = 40^\circ$, as the isocontour does not change in the
616 region $40^\circ < \beta < 80^\circ$, as discussed before. The most unstable wavenumber
617 increases up to 35% compared with its incompressible analogue. Also, the
618 compressibility induces a similar increase of $k_c/k_c^{Ma \rightarrow 0^+}$ and $k_{\max}/k_{\max}^{Ma \rightarrow 0^+}$,

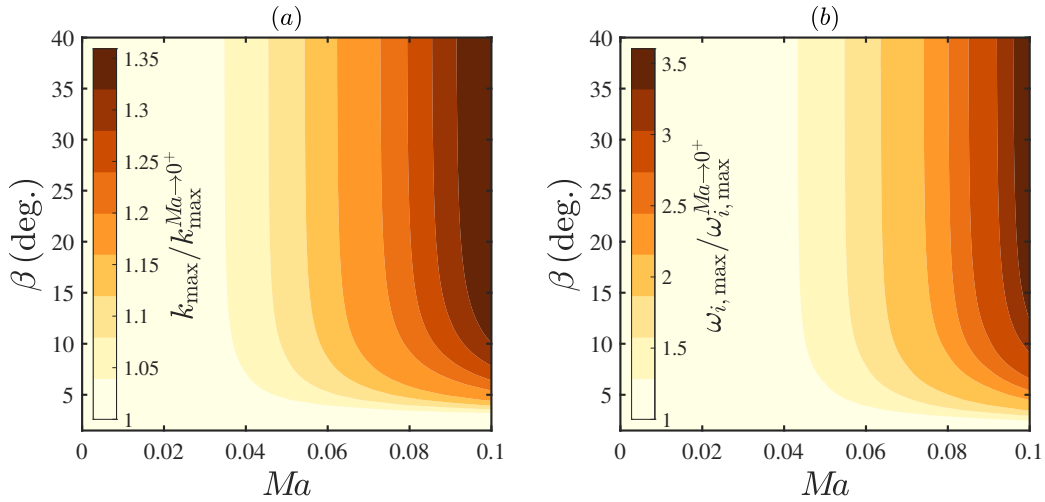


Figure 6: Effect of the Mach number $Ma = O(\varepsilon)$ and of the angle of inclination β on the stability of a falling water–glycerin film. Deviation of (a) the most unstable wavenumber k_{\max} from its incompressible limit $k_{\max}^{Ma \rightarrow 0^+}$, (b) the maximum growth rate $\omega_{i, \max}$ from its incompressible limit $\omega_{i, \max}^{Ma \rightarrow 0^+}$ for a value of Reynolds critical ratio (38) equal to $\text{RCR} = 1.05$. In overall terms, darker regions correspond to a greater destabilisation. The set of parameter values and fluid physical properties is the same specified for figure 3.

619 as shown in figures 6a and 5b, indicating that the destabilization involves
 620 both long and relatively short waves. Meanwhile, the maximum growth rate
 621 $\omega_{i, \max}$ can reach values up to about three and a half times higher than the
 622 incompressible one.

623 A method to explore the role of compressibility at highly-tilted config-
 624 urations consists in predetermining an adequate value of Re . For such a
 625 selection, we chose to cover a reasonably broad spectrum of slopes (with
 626 special attention to the steepest ones), without dropping the shallowness
 627 assumption.

628 Figure 7 displays the contours of the normalised cut-off wavenumber as

629 a function of the Mach number and of the inclination angle for two different
 630 fixed values of the Reynolds number, corresponding to (a) $Re = 1$ (with β
 631 ranging between 60° and 90°) and (b) $Re = 3$ (with $20^\circ \leq \beta \leq 60^\circ$). These
 632 combination of (Re, β) is such as to determine the onset of interfacial in-
 633 stability. From panels *a-b*, one may erroneously infer that, as β increases,
 634 $k_c/k_c^{Ma \rightarrow 0^+}$ exhibits a diminishing trend in contrast with previous results.
 635 However, this evolution is fully justifiable in the following terms: keeping
 636 Re fixed while the solid substrate steepens is tantamount to moving further
 637 away from the critical threshold, which corresponds to a progressive aug-
 638 mentation of the Reynolds critical ratio RCR, that is a situation where the
 639 compressibility-related effects on the destabilisation are less significant. As
 640 a consequence, figure 7 is consistent with what displayed in figures 3, 5 and,
 641 besides, helps in extending our analysis to the case of a vertical falling film
 642 flow.

643 As final part of the parametric study our sole aim is to investigate the
 644 influence of the Weber number We – and thus of the surface tension – on the
 645 compressibility-induced destabilising mechanism. To do so, we conclude by
 646 presenting numerical results for three different fluids: (i) water, (ii) aqueous
 647 solution of dimethylsulfoxide (DMSO), and (iii) aqueous solution of glycerin.
 648 As summarised in table 1, these fluids display different physical properties
 649 in terms of density, kinematic viscosity and surface tension, notwithstanding
 650 that the adopted barotropic EoS (14) remains unaltered among them. As
 651 regards the other variables belonging to the parameter space, the angle of
 652 inclination and the Reynolds critical ratio have been kept fixed and equal
 653 to $\beta = 15^\circ$ and $RCR = 1.05$, respectively. Such a choice corresponds to

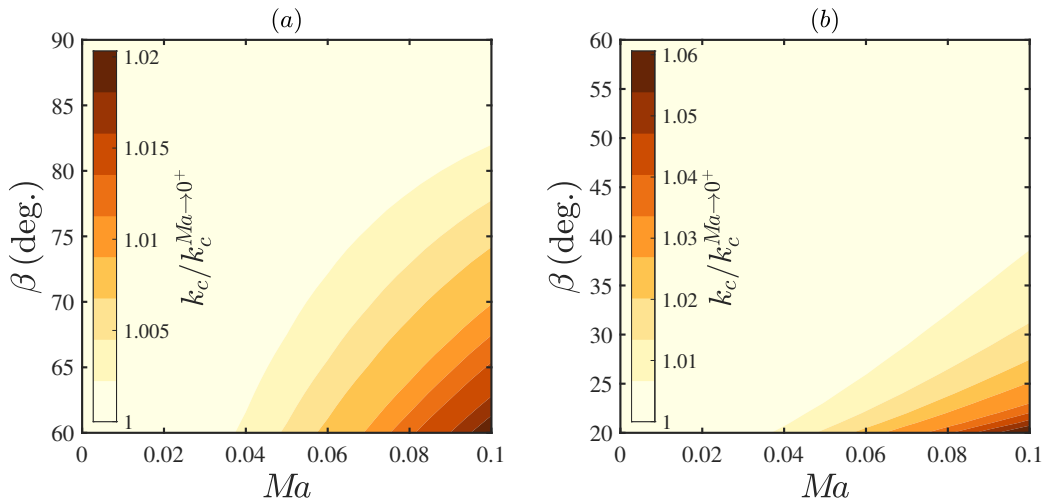


Figure 7: Effect of the Mach number $Ma = O(\varepsilon)$ and of the angle of inclination β on the stability of a falling water–glycerin film in terms of deviation of the cut–off wavenumber k_c from its incompressible limit $k_c^{Ma \rightarrow 0^+}$ with reference to the temporal growth rate of linear disturbances $k c_i(k)$, for two different fixed values of the Reynolds number, corresponding to (a) $Re = 1$ and (b) $Re = 3$. In overall terms, darker regions correspond to a greater destabilisation. The set of parameter values and fluid physical properties is the same specified for figure 3.

| Fluid | $\tilde{\rho}_0$ (kg m ⁻³) | $\tilde{\nu}_0$ (10 ⁻⁶ m ² s ⁻¹) | $\tilde{\gamma}_0$ (10 ⁻³ N m ⁻¹) | Ka |
|----------------|--|--|--|-------|
| Water | 1000.0 | 1.00 | 76.9 | 3592 |
| DMSO (83.11%) | 1098.3 | 2.85 | 48.4 | 509.5 |
| Glycerin (50%) | 1130.0 | 5.02 | 69.0 | 331.8 |

Table 1: Physical properties of fluids considered in the numerical stability calculations. The working liquids are the same as in Lavalle et al. (2019) (table 3 there): water, an aqueous solution of DMSO at 83.11% by weight, and an aqueous solution of glycerin at 50% by weight. The Kapitza number Ka is defined as $Ka = \tilde{\gamma}_0 \left(\tilde{\rho}_0 g^{1/3} \tilde{\nu}_0^{4/3} \right)^{-1}$, being $\tilde{\nu}_0 = \tilde{\mu}_0 / \tilde{\rho}_0$ the kinematic viscosity of the fluid under consideration.

654 the following set of values for the Weber number: (i) $We = 8.841 \cdot 10^{-4}$, (ii)
655 $We = 6.234 \cdot 10^{-3}$, (iii) $We = 1.156 \cdot 10^{-2}$. We have represented in figure 8
656 the cut-off wavenumber (a) and the maximum growth rate (b) as a function
657 of the Mach number Ma for the three liquids considered. As before, in
658 both panels the quantities shown are related to their analogues in the limit
659 of a perfectly incompressible flow. Within the present weakly compressible
660 scenario, we see that the onset of the long-wave instability is dimly affected
661 by surface tension and the destabilising effect of compressibility is felt earlier
662 at low Weber numbers.

663 6. Physical basis for the destabilising effect of compressibility

664 This section aims at clarifying the underlying physics behind the com-
665 pressibility effect on the onset of the flow primary instability.

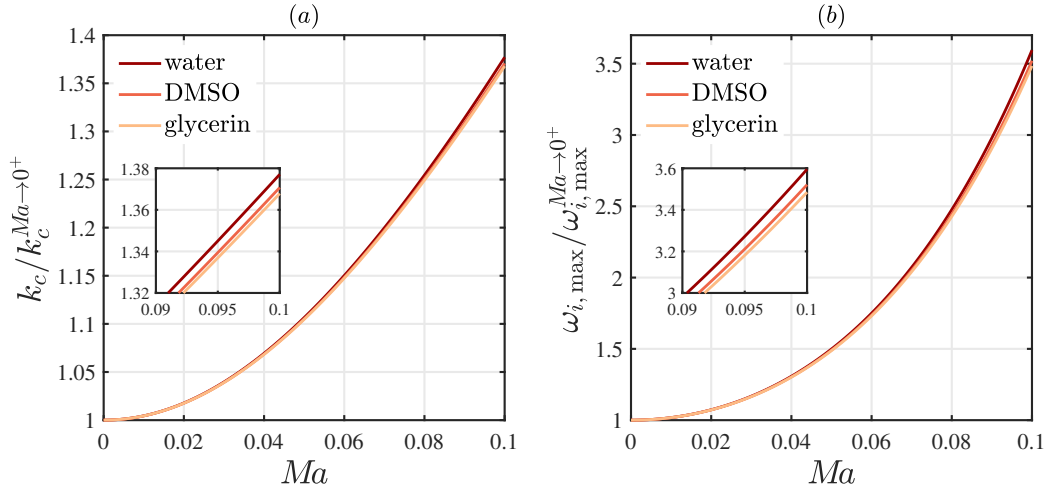


Figure 8: Effect of the Mach number $Ma = O(\varepsilon)$ on the stability of three falling film flows, each obtained employing one of the fluids detailed in table 1 in terms of physical properties and listed in the legend. Curves represent the deviation of (a) the cut-off wavenumber k_c from its incompressible limit $k_c^{Ma \rightarrow 0^+}$ and (b) the maximum growth rate $\omega_{i, \max}$ from its incompressible limit $\omega_{i, \max}^{Ma \rightarrow 0^+}$ with reference to the temporal growth rate of linear disturbances $\omega_i(k) \equiv k c_i(k)$, for a fixed value of the Reynolds critical ratio (38), equal to $\text{RCR} = 1.05$, and inclination angle $\beta = 15^\circ$.

666 *6.1. Whitham wave hierarchy*

667 The hydrodynamic stability of a shallow–water flow is linked to the prop-
 668 agation of interfacial waves (Whitham, 1974; Alekseenko et al., 1985, 1994;
 669 Ooshida, 1999; Kalliadasis et al., 2013). In this respect, Whitham’s theory
 670 of two–wave competition serves as a framework to interpret the linear sta-
 671 bility properties of the depth–averaged weakly–compressible model (25). To
 672 do so, we can make use of the dispersion relation (32) to study the mech-
 673 anism at the base of the compressible–induced destabilisation. Specifically,
 674 we formally recast (32) into the canonical form

$$i (c - c_k) + \Omega k (c - c_{d+})(c - c_{d-}) = 0, \quad (39)$$

675 where $c_k(k^2; Re, \beta, We, Ma)$, $c_{d\pm}(k^2; Re, \beta, We, Ma)$ and $\Omega(k^2; Re)$ are de-
 676 fined as follows

$$c_k = \frac{3}{2k^2 + 3} \left[3 + k^2 \left(-1 - \frac{4}{7} Re \cot \beta + \frac{24}{35} Re^2 - \frac{3}{Re} Ma^2 \cot \beta \right) - \frac{4}{21} \frac{Re^2}{We} k^4 \right] \quad (40a)$$

$$c_{d\pm} = -\frac{9 Ma^2 \cot \beta}{4 Re} \pm \frac{1}{2} \sqrt{\frac{4 k^2}{We} + \frac{12 \cot \beta}{Re} + \frac{108}{5} + \frac{81 Ma^4 \cot^2 \beta}{4 Re^2}} \quad (40b)$$

$$\Omega = \frac{Re}{2k^2 + 3}. \quad (40c)$$

677 Since the dispersion relation (39) recalls a two–wave structure, our reduced
 678 model (25) can be systematically reinterpreted as a second–order wave equa-
 679 tion

$$\underbrace{(\partial_t + c_k \partial_x) h}_{(i)} + \Omega \underbrace{(\partial_t + c_{d-} \partial_x)(\partial_t + c_{d+} \partial_x) h}_{(ii)} = 0, \quad (41)$$

680 which consists of two levels (i) (ii) of linear hyperbolic wave equations. The
 681 lower–order solutions to (i) are the *kinematic* waves since they origin from the

682 mass conservation (25a). These fast waves travel at a speed equal to c_k and
 683 they are dominant at long time and in the inertia-less limit $\Omega(Re) \rightarrow 0^+$.
 684 Conversely, the higher-order *dynamic* waves of the second kind (ii) arise
 685 from the film response, governed by the stress continuity condition (10b –
 686 10c) or – equivalently – by the momentum balance (25b), to variations in
 687 momentum, hydrostatic pressure and surface tension. They correspond to
 688 the limit $\Omega(Re) \rightarrow +\infty$. In their early stage, wavefronts located at the
 689 leading front and at the trailing edge of a produced wave packet begin to
 690 travel at a speed equal to c_{d^+} and c_{d^-} , respectively.

691 Interestingly, the dependence of $\Omega(k)$ on the wavenumber k is a mere con-
 692 sequence of the non-hyperbolicity of the evolution equation appertaining to
 693 the integral model (25), since terms whose order of spatial derivation exceeds
 694 the second would be ultimately included in it (Ruyer-Quil, 2012; Kalliadasis
 695 et al., 2013). Physically, this means that surface wave dispersion is modified
 696 by the streamwise viscous diffusion as early as the instability onset (Sharma
 697 and Dandapat, 2006). Anyway, $\Omega(k^2)$ is not appreciably affected by the
 698 squared wavenumber k^2 . In fact, by inspection of (40c), the denominator
 699 $2k^2 + 3 \approx 3$ within the long-wave limit ($k \ll 1$). As an *a posteriori* argu-
 700 ment, this fact adds legitimacy to the assumption of virtually non-dissipative
 701 fluid (Samanta et al., 2011), postulated in § 3.1 behind the adoption of (14)
 702 as barotropic EoS. The dependence (40a) of the kinematic wave speed c_k on
 703 the squared wavenumber k^2 gives an estimate of the dispersive role of the
 704 streamwise second-order viscous terms, sometimes referred to as “viscous
 705 dispersive effect” (Ruyer-Quil et al., 2008).

706 *6.1.1. Two-wave reframing of the critical threshold*

707 Whitham (1974) proved that the film primary instability can be precisely
708 reasoned in terms of competition between kinematic and dynamic waves.
709 Whenever a multi-speed equation of the kind given in (41) holds, long-wave
710 interfacial disturbances will damp on the condition that kinematic waves
711 travel at a speed ranging between the speeds of dynamic waves:

$$c_{d-} \leq c_k \leq c_{d+}. \quad (42)$$

712 The origin of the temporal stability criterion (42) stems from the evolution
713 of a localised precursory ripple (Ruyer-Quil, 2012). Since kinematic waves
714 tend to emerge from the wave packet at long times, whereas its short-term
715 dynamics is dominated by dynamic waves, the only stable situation is one
716 where the back and front of the wave travel at dynamic wave speed c_{d-} and
717 c_{d+} respectively, which implies constraint (42). The base state is marginally
718 stable if $c_{d-} = c_k$ or $c_{d+} = c_k$. Here, in practice, only the latter condition has
719 a binding character on the inception of the flow instability. Once evaluated
720 in the limit of infinitely long waves ($k \rightarrow 0^+$), it is verified that equality:

$$c_{d+} = c_k \quad (43)$$

721 is coherently able to recover (36), thus being in line with the expression for
722 the neutral stability threshold previously found by means of an asymptotic
723 expansion *à la* Yih (1963) for the wave celerity $c(k)$.

724 *6.1.2. Elucidation of the compressibility-induced destabilising effect*

725 To illustrate how compressibility enters Whitham's paradigm, we follow
726 the methodology adopted by Samanta et al. (2011) and Samanta (2014) for

727 liquid films falling along a slippery incline or in the presence of imposed shear
 728 stress, respectively. We consider the scenario discussed in figure 4(*a, d*), *i.e.*
 729 a water–glycerin film down a plane inclined at $\beta = 1.5^\circ$ and $\beta = 12^\circ$. For
 730 these two angles of inclination, figure 9 compares the kinematic wave speed
 731 c_k and the dynamic one c_{d+} given by (40a) and (40b) as a function of the
 732 squared dimensionless wavenumber k^2 , both within the incompressible limit
 733 $Ma \rightarrow 0^+$ (solid lines) and in a slightly compressible case, where $Ma = 0.1$
 734 (dashed lines). Two values of the Reynolds critical ratio RCR – beyond
 735 the stability threshold, though in its vicinity – have been examined: (*a, c*)
 736 $\text{RCR} = 1.025$ and (*b, d*) $\text{RCR} = 1.075$.

737 Figure 9 evidences that the compressibility contributes in lowering both
 738 the dynamic and the kinematic wave speeds. For further clarification, fig-
 739 ure 9 has been completed with a proper close–up of the plane portion where
 740 curves cross each other. One easily realizes that each compressible cut–off
 741 point (void circle) is always located at a higher squared wavenumber k^2 in
 742 comparison with its incompressible analogue (filled circle).

743 The kinematic wave speed c_k , however, is much less affected by the com-
 744 pressibility than the dynamic one c_{d+} . This can be inspected by a brief dis-
 745 cussion on the role of inertia. Let us first consider the low–angle configuration
 746 (upper panels). In such a scenario, the variation in the Reynolds critical ratio
 747 RCR in figure 9*a, b* seems to only have a minor impact on the compressible
 748 dynamic celerity c_{d+} in terms of vertical shift. On the other hand, for the
 749 greatest RCR (panel *b*), the parabolic–like trend of the kinematic celerity c_k
 750 evolves with respect to k^2 in such a way that its descending tract gets drasti-
 751 cally steeper in the vicinity of its point of intersection with the graph of the

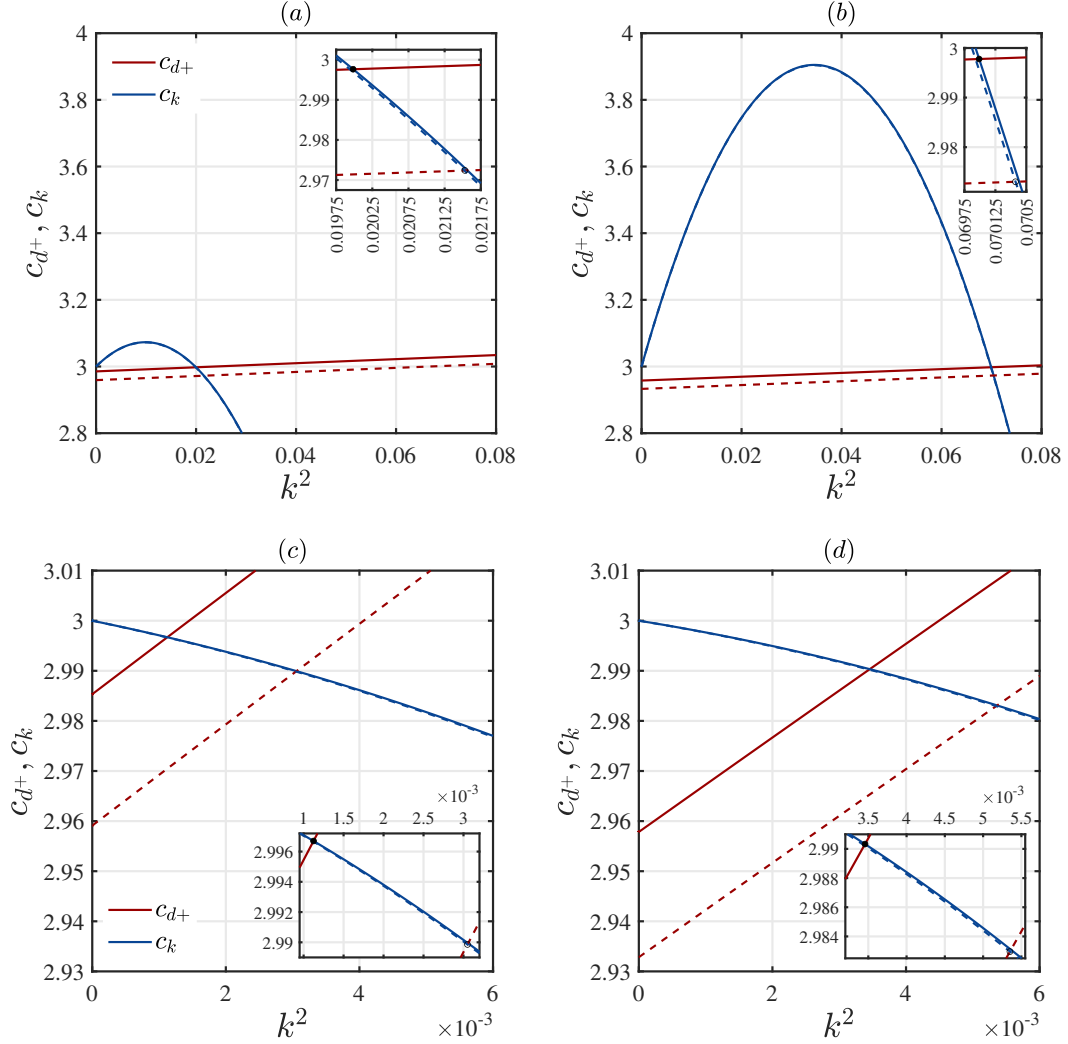


Figure 9: The variation of dynamic c_{d+} (in red) and kinematic c_k (in blue) wave speeds as a function of k^2 when the Mach number Ma passes from zero (solid lines) to a value of 0.1 (dashed lines), for different configurations in terms of angle of inclination β and Reynolds critical ratio RCR. (a-b): $\beta = 1.5^\circ$. (c-d): $\beta = 12^\circ$. Left panels: RCR = 1.025. Right panels: RCR = 1.075.

752 dynamic wave velocity c_{d+} . As a consequence, the compressibility-induced
 753 destabilisation gets noticeably reduced when the Reynolds number increases.

754 We close this section by comparing panels ($c - d$), for which the incli-
 755 nation angle is $\beta = 12^\circ$. Here we notice that the speed of kinematic waves
 756 c_k is less sensitive in comparison with the previous low-angle configuration
 757 to the same increase in the Reynolds critical ratio, from $\text{RCR} = 1.025$ (left)
 758 to $\text{RCR} = 1.075$ (right). Meanwhile, the dynamic wave speed c_{d+} , which
 759 increases as a straight line with the square of the wavenumber k^2 , under-
 760 goes deceleration by enhancing compressibility, but also by increasing RCR,
 761 leading to an attenuation of the compressibility-induced destabilisation.

762 *6.2. Impact of compressibility on flow-related quantities*

763 Aiming at finding a physical source to which the overflow uncovered in
 764 § 4.1.1 may be attributed, we rephrase the pertinent perturbative analogue
 765 $\Delta q_{\text{rel}}^{(2)} = (q^{(2)} - q^{(2)}|_{\text{Ma} \rightarrow 0+}) / q^{(0)}$ in terms of dimensional variables, which
 766 gives:

$$\Delta q_{\text{rel}}^{(2)} = \frac{\Lambda g \tilde{h} \cos \beta}{8 \tilde{a}_0^2}. \quad (44)$$

767 In a similar way, as the leading-order wall shear stress $\tau_w^{(0)} \equiv \partial_y u^{(0)}|_{y=0}$ is
 768 employed as normalising quantity for the extra wall shear stress profile, we
 769 obtain:

$$\Delta \tau_{w, \text{rel}}^{(2)} = \frac{\Lambda g \tilde{h} \cos \beta}{6 \tilde{a}_0^2}. \quad (45)$$

770 The same functional form is manifestly shared by (44) and (45). A simple
 771 physical interpretation of the ratio therein contained, namely $g \tilde{h} \cos \beta / \tilde{a}_0^2$,
 772 can be given in the following terms:

$$\Delta q_{\text{rel}}^{(2)}, \Delta \tau_{\text{rel}}^{(2)} \propto \frac{\tilde{\rho} g \tilde{h} \cos \beta}{\tilde{\rho} \tilde{a}_0^2} = \frac{\tilde{P}_{\text{h}}^{\text{eff}}}{\tilde{P}_{\text{a}}}. \quad (46)$$

773 We can notice that (46) accounts for the ratio between the effective com-
774 ponent of the hydrostatic pressure \tilde{P}_h^{eff} exerted along the cross-stream di-
775 rection by the wavy fluid column of height \tilde{h} , as stipulated by Stevin’s law,
776 and a reference acoustic pressure \tilde{P}_a . As a matter of fact, the whole operat-
777 ing mechanism through which compressibility acts as a destabilising factor
778 for the temporal development of long-wave linear disturbances should be
779 intended as the competition of multiple effects: for decreasing angles of in-
780 clination, the gravitational effect is emphasised as $\cos \beta$ increases, but such a
781 trigger for destabilisation is counterbalanced by the decrease of the uniform
782 film thickness \tilde{h}_N , which is a function of $\sin \beta$, and so of \tilde{h} .

783 *6.2.1. Compressible lag of flow rate perturbations*

784 In order to explain the physical mechanism responsible for the compressibility-
785 induced flow destabilisation, we adapt the basic rationale behind the method-
786 ology followed by Lavalle et al. (2019) in the context of confined falling liquid
787 films in presence of an active upper phase. We start by recalling that the
788 driving mechanism of Kapitza instability can be traced back to inertia, which
789 is responsible for the time lag between the actual liquid flow rate $q(h(x, t))$
790 and its inertialess target value:

$$q^*(h(x, t)) = \underbrace{\frac{\Lambda h^3}{3}}_{q^{*,g}} + \underbrace{\frac{Ma^2 h^4 \Lambda^2 \cot \beta}{8 Re}}_{q^{*,Ma}} + O(\varepsilon^2). \quad (47)$$

791 Here the second-order contribution arising from the flow compressibility has
792 been highlighted individually, without expressly taking its limit as $Re \rightarrow 0^+$
793 owing to its divergent behaviour. Instead, two other Re -independent second-
794 order terms contained within the expression of $q^{(2)}$ and arising in particular

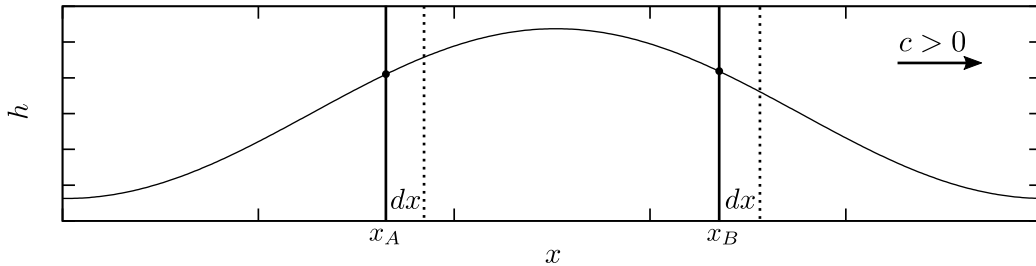


Figure 10: Description scheme of the inertia-based mechanism of the Kapitza instability: by comparison between two points of abscissa x_A and x_B , located at opposite sides of a wave peak, the local film flow rate $q(x, t)$ is delayed in accommodating itself to film thickness variations induced by the passage of the superficial disturbance of speed c .

795 from the normal stress continuity condition (10b) at order $O(\varepsilon^2)$ have not
 796 been explicitly written in (47) and disregarded for simplicity in subsequent
 797 calculations. Such a decomposition therefore appears to be accurate at $O(\varepsilon)$
 798 and it is used only as a means to gain insight at the mechanism at work
 799 by estimating the relative importance of each individual component in the
 800 destabilisation of the weakly-compressible flow.

801 The destabilising role of inertia on single-peaked Kapitza waves can be
 802 explained resorting to the analysis followed by Dietze (2016), who considered
 803 the history of two points located along the film free-surface either side of a
 804 wave crest. With reference to figure 10, at the abscissa x_B upstream of the
 805 wave hump, where $\partial_x h < 0$, the film thickness increases in time as the wave
 806 covers a distance dx , and so does the flow rate q along the x direction, in
 807 accordance with Benney's leading-order asymptotic expansion (B.1c). Con-
 808 versely, at the abscissa x_A downstream of the wave hump, the film thickness
 809 and the flow rate decrease when the wave covers dx . In the presence of in-
 810ertia, the flow rate cannot adapt instantaneously to such a film thickness

811 variation. As a result, the flow rate in x_A will be too high while it will be too
 812 low in x_B . The ensuing discrepancy in flow across the wave peak accounts
 813 for its growth. Such a response is more intense as the lag phase of the actual
 814 flow rate q behind its target value q^* increases.

815 According to (47), the effect of gravity through the cubic dependence
 816 of $q^{*,g}(h)$ on h tends to promote variations in q^* between the wave hump
 817 and the wave trough as an outcome of the change in film thickness h . The
 818 non-negative compressible contribution $q^{*,Ma}(h)$ exacerbates such an effect,
 819 increasingly so as the corresponding term in (47) gains relevance. For a per-
 820 tinent quantification, variables appearing in equation (47), *viz.* the wavy film
 821 thickness h and the inertialess film flow rate q^* , are linearly perturbed around
 822 the aforesaid (see § 4.1.1) base state vector \mathbf{Q}_0 , via superimposition of
 823 infinitesimal disturbances of amplitude $\|\hat{\mathbf{Q}}\| \ll \|\mathbf{Q}_0\|$:

$$h(x, t) = h_0 + \hat{h}(x, t) \quad (48a)$$

$$q^*(h) = q_0 + \hat{q}(h). \quad (48b)$$

824 By virtue of (48) it is now possible to discriminate between the magnitude
 825 of perturbations \hat{q}^g and \hat{q}^{Ma} , which are, respectively, of gravitational and
 826 compressible provenance:

$$\hat{q}(\hat{h}) = \underbrace{\Lambda h_0^2 \hat{h}}_{\hat{q}^g} + \underbrace{\frac{Ma^2 h_0^3 \hat{h} \Lambda^2 \cot \beta}{2 Re}}_{\hat{q}^{Ma}}. \quad (49)$$

827 The following expression can be obtained for the so-defined compressible-
 828 to-total amplitude ratio \hat{q}^{Ma}/\hat{q} :

$$\left| \frac{\hat{q}^{Ma}}{\hat{q}} \right| = \frac{3 Ma^2 \cot \beta}{2 Re + 3 Ma^2 \cot \beta} \stackrel{(*)}{=} \frac{9 Ma^2}{5 RCR + 9 Ma^2}, \quad (50)$$

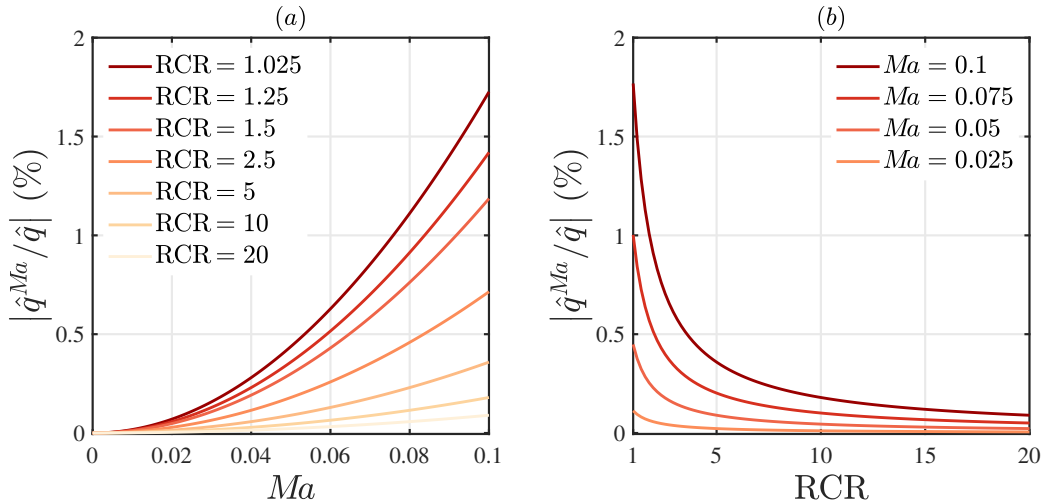


Figure 11: Percentage contribution of the compressibility-related perturbation \hat{q}^{Ma} to the total inertialess flow rate perturbation \hat{q} (49) (a) as a function of the Mach number $Ma = O(\varepsilon)$ for different fixed values of the Reynolds critical ratio RCR (displayed in the legend) and (b) *vice versa*.

829 in which use has been made of the equality $\Lambda = 3$ and of the identity $h_0 \equiv 1$,
830 (*) together with the definition of the Reynolds critical ratio RCR (38), cou-
831 pled with the incompressible evaluation of the critical threshold $\lim_{Ma \rightarrow 0^+}$ (36),
832 in lieu of the Reynolds number Re . Figure 11 shows that the ratio expressed
833 by (50) (a) increases with the Mach number Ma and (b) decreases with the
834 Reynolds critical ratio RCR, which is in accordance with the most promi-
835 nent role played by compressibility in the film flow destabilisation shown in
836 section § 5.

837 7. Conclusions

838 Liquid films occur over a wide range of length scales and are central
839 to numerous areas of pure and applied sciences (Craster and Matar, 2009).

840 The development of long-wave instabilities along its interface leads to self-
841 excitation of non-trivial dynamics (Sharma and Dandapat, 2006). The mo-
842 tivation behind this study is addressing theoretically how changes in the
843 fluid density fit into this context. For such purpose, we have discussed three
844 guiding questions.

845 (i) How does compressibility affect the structure of a depth-integral model?

846 We considered a barotropic relation involving the Mach number of the mean
847 flow. Under the assumption of weak compressibility $Ma \ll 1$, the density
848 of the fluid is found to be exponentially stratified against gravity along the
849 crosswise direction. In the final depth-averaged system (25) this is reflected
850 in two additional terms: one $\propto \cot \beta Ma^2/Re$ in the continuity equation, and
851 the other $\propto \cot \beta Ma^2/Re^2$ in the momentum conservation equation.

852 (ii) To what extent does compressibility take part in long-wave insta-
853 bility? According to our linear analysis, a low degree of compressibility
854 boosts the inception of interfacial instability. This effect is most marked in
855 low-inertial regimes. For instance, with reference to figure 5*b*, the instabil-
856 ity threshold of a water-glycerin film flow having $Re = 2.40$, $\beta = 20^\circ$ and
857 $Ma = 0.1$ as set of distinctive parameters is seen to increase by 35% in terms
858 of the cut-off wavenumber with respect to its incompressible analogue. A
859 higher-order additive correction of the base flow rate, hydrostatic in nature,
860 has been highlighted via (44). As perspective on future research, the de-
861 rived depth-integrated model (25) will be also of interest to simulate the
862 non-linear dynamics of weakly-compressible falling liquid films, on condi-
863 tion that proper manipulations are performed for the numerical treatment of
864 capillary terms (Lavalle et al., 2015).

865 (iii) Which is the underlying physical foundation? Albeit of small magni-
866 tude, differences between the compressible and incompressible nature of the
867 long-wave instability can be traced back to a compressible-induced deceler-
868 ation of dynamic waves (figure 9) or, equivalently, to an additional inertia-
869 induced delay (relative to the kinematic waves) of the flow rate in adapting
870 to a time-varying film-thickness (figure 11).

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876 Declaration of interests

877 The authors report no conflict of interest.

878 Appendix A. Reduction of the pressure profile

879 The complete solution of (16) is given by:

$$\begin{aligned}
 p(x, y, t; \vartheta) = & p_i + \frac{\overbrace{\exp\left[\frac{\cos\beta}{Fr}Ma^2(h-y)\right]^{-1}}^{\diamond}}{Ma^2} - \frac{\varepsilon^2}{We}\partial_{xx}h + \\
 & + \frac{\varepsilon}{Re}\left[\mathcal{W} - \mathcal{W}|_h \overbrace{\exp\left[\frac{\cos\beta}{Fr}Ma^2(h-y)\right]}^{\diamond} - \left(\frac{2}{3} - \vartheta\right)(\partial_x u + \partial_y v)|_h \right. \\
 & \left. + 2(\partial_y v)|_h - 2\overbrace{\partial_x h(\partial_y u)|_h}^{\diamond}\right] - \mathcal{I}(y, \mathcal{W}) \exp\left(-\frac{\cos\beta}{Fr}Ma^2 y\right),
 \end{aligned}
 \tag{A.1}$$

880 where $\mathcal{I}(y, \mathcal{W})$ is the so-defined primitive

$$\mathcal{I}'(y, \mathcal{W}) = \varepsilon Ma^2 \frac{\cos \beta}{Re Fr} \mathcal{W} \exp\left(\frac{\cos \beta}{Fr} Ma^2 y\right), \quad (\text{A.2})$$

881 the prime mark denoting total differentiation with respect to y . By inspection
 882 of (A.2), since it is assumed $Re \sim Fr = O(1)$, \mathcal{I}' can be regarded as an
 883 $O(\varepsilon Ma^2)$ residual contribution, originating from the process of integration
 884 by parts in the context of the application of Duhamel's technique. Given that
 885 its analytical integration would at least require *a priori* knowledge concerning
 886 the explicit expression for the unknown spatial derivatives of the velocity field
 887 $\mathbf{v} = (u, v)$ involved within \mathcal{W} as part of the integrand function (A.2), we seek
 888 for a low-compressibility restriction of the kind

$$\varepsilon Ma^2 \lesssim \varepsilon^3, \quad (\text{A.3})$$

889 a condition wherein it is legitimate to consistently ignore its respective con-
 890 tribution within the ultimate problem (9) via (14). Indeed, assignment (A.3)
 891 has been formalised in asymptotic terms through the equivalence relation (19),
 892 with $\alpha \geq 1$ and $M = O(1) \in \mathbb{R}_0^+$. By recalling expansion (20) with (A.3)
 893 in mind, the $O(\varepsilon)$ -exponential term denoted as \blacklozenge can be shortened to the
 894 unitary value only. Furthermore, starting from the definition of \mathcal{W} – jointly
 895 given with (16) – it is straightforward to verify that

$$-\mathcal{W}|_h - \left(\frac{2}{3} - \vartheta\right) (\partial_x u + \partial_y v)|_h + 2 (\partial_y v)|_h \equiv -(\partial_x u)|_h. \quad (\text{A.4})$$

896 Finally, the boundary condition (10c) highlights the fact that $(\partial_y u)|_h =$
 897 $O(\varepsilon^2)$, thereby allowing for the removal of \blacklozenge , which ultimately contributes
 898 as an $O(\varepsilon^3)$ term within (A.1). As a result, (A.1) is consistently tantamount
 899 to (17).

900 **Appendix B. Asymptotic expansions**

901 *Appendix B.1. Leading order $O(\varepsilon^0)$*

$$u^{(0)}(h(x, t), y) = -\frac{\Lambda(y^2 - 2hy)}{2} \quad (\text{B.1a})$$

$$v^{(0)}(h(x, t), y) = -\frac{\Lambda(\partial_x h) y^2}{2} \quad (\text{B.1b})$$

$$q^{(0)}(h(x, t)) = \frac{\Lambda h^3}{3} \quad (\text{B.1c})$$

902 The steady-state flat-film solution, corresponding to Nusselt flow (Nusselt,
 903 1916), can be recovered by substituting unity for h in equations (B.1). This
 904 shows that the leading order of Benney's development corresponds to local
 905 equilibrium.

906 *Appendix B.2. First order $O(\varepsilon^1)$*

$$u^{(1)}(h(x, t), y) = \frac{Re \varepsilon^2}{We} \partial_{xxx} h \left(hy - \frac{y^2}{2} \right) + \frac{Re \cos \beta}{Fr} \partial_x h \left(-hy + \frac{y^2}{2} \right) +$$

$$+ Re \Lambda^2 \partial_x h \left(\frac{hy^4}{24} - \frac{h^4 y}{6} \right) + Re \Lambda \partial_t h \left(\frac{y^3}{6} - \frac{h^2 y}{2} \right) \quad (\text{B.2a})$$

$$v^{(1)}(h(x, t), y) = \frac{Re \varepsilon^2}{We} \left[\partial_{4x} h \left(\frac{y^3}{6} - \frac{hy^2}{2} \right) - (\partial_x h) (\partial_{xxx} h) \frac{y^2}{2} \right] +$$

$$+ \frac{Re \cos \beta}{Fr} \left[\partial_{xx} h \left(-\frac{y^3}{6} + \frac{hy^2}{2} \right) + \partial_x^2 h \frac{y^2}{2} \right] +$$

$$+ Re \Lambda y^2 \left[\partial_{tx} h \left(-\frac{y^2}{24} + \frac{h^2}{4} \right) + (\partial_t h) (\partial_x h) \frac{h}{2} \right] +$$

$$+ Re \Lambda^2 \left[\partial_{xx} h \left(-\frac{hy^5}{120} + \frac{h^4 y^2}{12} \right) + \partial_x^2 h \left(-\frac{y^5}{120} + \frac{h^3 y^2}{3} \right) \right] \quad (\text{B.2b})$$

$$\begin{aligned}
u^{(2)}(h(x, t), y) &= \frac{Re^2 \Lambda \varepsilon^2}{We} \partial_{xxxx} h \left(\frac{y^6}{360} - \frac{hy^5}{60} + \frac{h^2 y^4}{12} - \frac{h^3 y^3}{6} + \frac{7h^5 y}{30} \right) + \\
&+ \frac{Re^2 \Lambda \varepsilon^2}{We} \partial_x h \partial_{xxx} h \left(\frac{5hy^4}{12} - \frac{3h^2 y^3}{2} + \frac{17h^4 y}{6} \right) + \\
&+ \frac{Re^2 \Lambda \varepsilon^2}{We} \partial_{xx}^2 h \left(\frac{hy^4}{4} - h^2 y^3 + 2h^4 y \right) + \\
&+ \frac{Re^2 \Lambda \varepsilon^2}{We} \partial_x^2 h \partial_{xx} h \left(\frac{y^4}{2} - 2hy^3 + 4h^3 y \right) + \\
&+ \frac{Re^2 \Lambda \cos \beta}{Fr} \partial_x^2 h \left(-\frac{hy^4}{6} + \frac{h^2 y^3}{2} - \frac{5h^4 y}{6} \right) + \\
&+ \frac{Re^2 \Lambda \cos \beta}{Fr} \partial_{xx} h \left(-\frac{y^6}{360} + \frac{hy^5}{60} - \frac{h^2 y^4}{12} + \frac{h^3 y^3}{6} - \frac{7h^5 y}{30} \right) + \\
&+ \Lambda \partial_{xx} h \left(-\frac{y^3}{3} - \frac{hy^2}{2} + \frac{5h^2 y}{2} \right) + \\
&+ Re^2 \Lambda^3 \partial_{xx} h \left(-\frac{hy^8}{4480} + \frac{h^2 y^7}{560} - \frac{h^3 y^6}{180} + \frac{h^4 y^5}{120} + \frac{h^5 y^4}{72} - \frac{h^6 y^3}{18} + \frac{29h^8 y}{315} \right) + \\
&+ Re^2 \Lambda^3 \partial_x^2 h \left(-\frac{y^8}{4480} + \frac{hy^7}{560} - \frac{7h^2 y^6}{720} + \frac{h^3 y^5}{30} + \frac{5h^4 y^4}{72} - \frac{h^5 y^3}{3} + \frac{38h^7 y}{63} \right) + \\
&+ \Lambda \partial_x^2 h \left(5hy - \frac{y^2}{2} \right) + \frac{M^2 \Lambda \cos \beta}{Fr} \left(\frac{y^3}{6} - \frac{hy^2}{2} + \frac{h^2 y}{2} \right)
\end{aligned} \tag{B.3}$$

913 **Appendix C. Validation with Orr–Sommerfeld problem within the**
914 **incompressible limit**

915 Our second–order model (25) correctly recovers the expressions for $c^{(0)}$, $c^{(1)}|_{Ma \rightarrow 0^+}$
916 and $c^{(2)}|_{Ma \rightarrow 0^+}$, which show accordance with the asymptotic expansions of
917 solutions to Orr–Sommerfeld boundary–value problem – reported in Ruyer–
918 Quil and Manneville (1998). However, it is expected that higher–order ex-

| $Re \cot^2 \beta$ | $Re^2 \cot \beta$ | $\cot \beta$ | Re/We | Re^3 | Re |
|-------------------|-------------------|--------------|---------|--------|------|
| -16.7 | 21.6 | -63.0 | 0 | 28.9 | 7.8 |

Table C.2: Percent errors [%] (expressed to one decimal place) committed by the incompressible evaluation of the present second-order model (25) $_{|Ma \rightarrow 0^+}$ in the estimate of polynomial coefficients \star of the $O(k^3)$ incompressible wave celerity $c^{(3)}_{|Ma \rightarrow 0^+}$, given by (34d) $_{|Ma \rightarrow 0^+}$ by comparison with the exact ones (Ruyer-Quil and Manneville, 1998; Chang and Demekhin, 2002) provided by the Orr-Sommerfeld theory.

919 pressions of $c^{(j)}$ with $j > 2$ are not correctly captured. Specifically, when the
920 incompressible limit of $c^{(3)}$, expressed by (34d) $_{|Ma \rightarrow 0^+}$, is contrasted with its
921 exact Orr-Sommerfeld (O-S) analogue, we notice that all terms are present,
922 but with different numerical coefficients in front of them in almost every oc-
923 currence \star . As shown in table C.2, such discrepancies can be quantified in
924 terms of relative percentage deviation

$$\frac{\star_{(34d)}^{(3)}_{|Ma \rightarrow 0^+} - \star_{\text{O-S}}^{(3)}}{\star_{\text{O-S}}^{(3)}} \cdot 100\% \quad [\%]. \quad (\text{C.1})$$

925 A numerical validation of the present second-order weakly-compressible
926 model within its incompressible limit (25) $_{|Ma \rightarrow 0^+}$ can be accomplished by
927 comparing its predictions to data from the literature concerning the long-
928 wave interfacial instability for a liquid falling film flow. Figure C.12 com-
929 pares growth rate and angular frequency of linear surface waves with results
930 of Brevdo et al. (1999) for the case of a liquid film falling down an incline
931 within a passive atmosphere. We remark that agreement is achieved between
932 the two sets of data with reference to the immediate proximity to the limit of
933 infinitely long-wave ($k \rightarrow 0^+$), as long as the Reynolds number Re is chosen

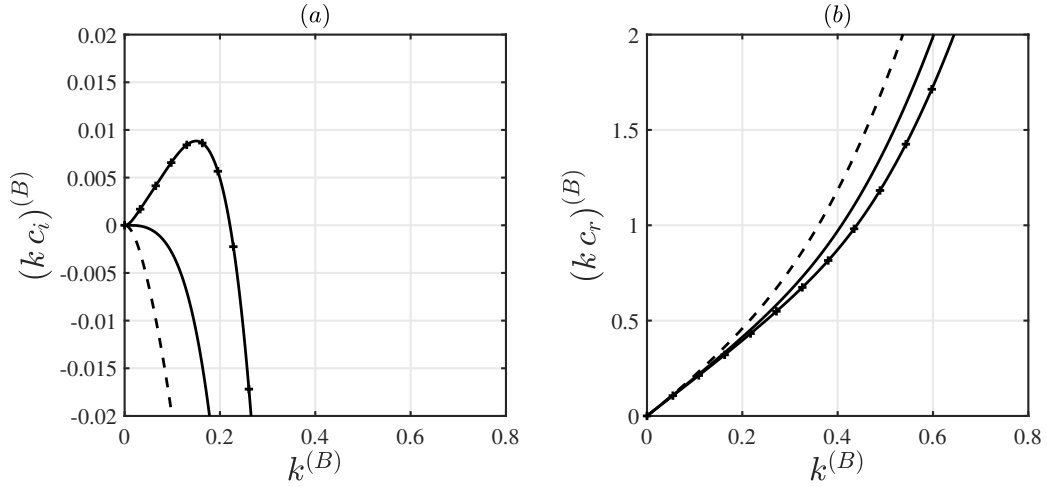


Figure C.12: Comparison of the dimensionless (a) temporal growth rate $k c_i(k)$ and (b) angular wave frequency $k c_r(k)$ between our work and Brevdo et al. (1999) (figure 2 there). Parameter values: $g = 9.81 \text{ m s}^{-2}$, $\beta = 4.6^\circ$, $\tilde{\rho}_0 = 1130 \text{ kg m}^{-3}$, $\tilde{\mu}_0 = 5.673 \cdot 10^{-3} \text{ Pa s}$, $\tilde{\gamma}_0 = 69.0 \cdot 10^{-3} \text{ N m}^{-1}$. Values of the Reynolds number $Re^{(B)} = (3/2) Re$ according to Brevdo's scaling: 10 (dashed line), $Re_{cr}^{(B)} = (5/4) \cot \beta$ (bare solid line), 20 (pluses). Note that $k^{(B)}$, $(k c_r)^{(B)}$ and $(k c_i)^{(B)}$ are scaled as in Brevdo et al. (1999), *i.e.* using the Nusselt film thickness \tilde{h}_N and the free-surface velocity $(3/2) \tilde{U}_N$ as length and velocity scales, respectively, instead of the film mean velocity \tilde{U}_N as done here.

934 to be compliant with the pertinent assumption $Re = O(1)$ made in § 3.

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