Higher-order asymptotic model for a heterogeneous beam, including corrections due to end effects

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The study presented here is devoted to the 1D-modeling of a transversely heterogeneous beam with an arbitrary cross-section. The formal asymptotic expansion method is used, so that the initial 3D-problem splits in a sequence of 2D-problems, posed on the cross-section, and 1D-problems, which give the governing equations of the overall outer expansion. However, considering the higher-order terms of this expansion will actually improve the approximation of the 3D solution provided that edge effects are taken into account. The latter are treated following a decay analysis technique, which provide the boundary conditions of the 1D-problems in such a way that the edge effects decay rapidly. Moreover, it is shown that accounting for end effects exempts from using a refined model, since the governing equations for the overall full outer expansion correspond to the classical Euler-Bernoulli ones. The example of a cantilevered layered sandwich beam is treated and results obtained prove that the method enables to recover the exact 3D interior solution, with a very good accuracy.

Introduction

Several approaches have been proposed for the 1D-modeling of heterogeneous and anisotropic beams, accounting for non-classical effects such as the transverse shear and transverse normal deformation, and cross-sectional warping.
Two general approaches can be identified : the method of hypotheses and the formal asymptotic expansion technique. The first approach is based on introducing ad hoc assumptions regarding the displacements, strains or stresses distribution in the cross-section, and many higher-order theories have been discussed in the literature (see references in[1] for example). By contrast, the asymptotic expansion method is free of a priori assumptions, and enables the *construction* of higher-order theories through the computation of the successive terms of the expansion. Moreover, this method is particularly well-suited for taking into account edge effects, which have to be incorporated in any refined theory to actually improve the approximation of the 3D solution[2,3].

In this paper, the asymptotic expansion method is applied to the case of a transversely heterogeneous beam, with an arbitrary cross-section. In a first step, the outer expansion is studied : the 3D elasticity problem splits in a sequence of 2D-problems, posed on the cross-section, and 1D differential equations, which provide the overall beam response. The boundary conditions of these 1D-problems are then derived following a decaying state analysis[4]. The example of a cantilevered layered sandwich beam is treated and results obtained, compared to the 3D solution, prove the effectiveness and the reliability of the method.

The asymptotic outer solution

The initial three dimensional problem

The 3D slender structure considered herein is a prismatic rod, of axis $x_\alpha = 0$, which occupies the configuration $\Omega^\epsilon = S^\epsilon \times [0, L]$, see Fig. 1. This structure possesses a geometrical small parameter ε, defined as the ratio of a cross-sectional length scale of the beam to the axial one. The cross-section S^ϵ may be heterogeneous, and of any arbitrary shape (the cases of a solid, closed or open cross section can be indifferently studied). The boundary of Ω^ε is defined by $\partial\Omega^\varepsilon = \Gamma^\varepsilon \cup \Gamma_0^\varepsilon \cup \Gamma_L^\varepsilon$, with $\Gamma_0^\varepsilon = S^\epsilon \times \{0\}$ and $\Gamma_L^\varepsilon = S^\epsilon \times \{L\}$ the two end-sections of the beam, and Γ^ε the cross-section boundary. For later simplification, the beam is

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considered to be free of body forces, as well as surface tractions on Γ^ε. The left end Γ_0^ε is clamped and stress data $\bar{\sigma}_{3i}^\varepsilon(x_1, x_2)$ are prescribed on the right end Γ_L^ε.

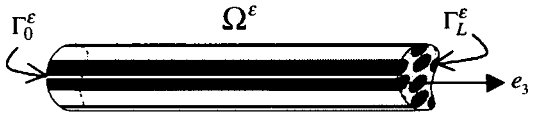

Fig. 1 3D slender heterogeneous structure Ω^ε

The 3D static problem P^ε of linear elasticity consists in finding the fields $\boldsymbol{\sigma}^\varepsilon$, $\mathbf{e}^\varepsilon$ and $\mathbf{u}^\varepsilon$, such that:

$$\left\{\begin{array}{l} \mathbf{div_x}\boldsymbol{\sigma}^\varepsilon = \mathbf{0} \\ \boldsymbol{\sigma}^\varepsilon = \mathbf{a}^\varepsilon(x_1, x_2) : \mathbf{e}^\varepsilon(\mathbf{u}^\varepsilon) \\ \mathbf{e}^\varepsilon(\mathbf{u}^\varepsilon) = \mathbf{grad_x^s}(\mathbf{u}^\varepsilon) \\ \boldsymbol{\sigma}^\varepsilon.\mathbf{n} = \mathbf{0} \ \ on \ \ \Gamma^\varepsilon \\ \boldsymbol{\sigma}^\varepsilon.\mathbf{e_3} = \bar{\sigma}_{3i}^\varepsilon(x_1, x_2).\mathbf{e_i} \ \ on \ \ \Gamma_L^\varepsilon \\ \mathbf{u}^\varepsilon = \mathbf{0} \ \ on \ \ \Gamma_0^\varepsilon \end{array}\right. \tag{1}$$

where $\mathbf{a}^\varepsilon(x_1, x_2)$ denotes the elastic moduli tensor, and where $\mathbf{grad_x^s}$ and $\mathbf{div_x}$ correspond respectively to the symmetric strain and divergence operators, with respect to the spatial coordinate $\mathbf{x}$.

The asymptotic expansion method

The first step of the method consists in defining a problem equivalent to (1), but now posed on a fixed domain which does not depend on the small parameter ε.
To this end, we introduce the following change of variable[5,6] to take into account the slenderness of the beam cross-section :

$$(y_1, y_2, z_3) = (\frac{x_1}{\varepsilon}, \frac{x_2}{\varepsilon}, x_3) \tag{2}$$

Consequently, z_3 represents the *slow* or *large scale* or *macroscopic* 1D-variable of the problem and $y_\alpha = \frac{x_\alpha}{\varepsilon}$ the *fast* or *small scale* or *microscopic* 2D-one.
Throughout this paper, Latin indices take values in the set {1,2,3} while Greek indices in {1,2}. We also use the Einstein summation convention on repeated indices.

According to this change of variable, we associate the new strain and divergence operators in the following manner :

$$\left\{\begin{array}{l} \mathbf{grad_x^s}. = \mathbf{grad}_{z_3}^\mathbf{s}. + \frac{1}{\varepsilon}\mathbf{grad}_{y_\alpha}^\mathbf{s}. \\ \mathbf{div_x}. = \mathbf{div}_{z_3}. + \frac{1}{\varepsilon}\mathbf{div}_{y_\alpha}. \end{array}\right. \tag{3}$$

where $\mathbf{grad}_{z_3}^\mathbf{s}$ and $\mathbf{div}_{z_3}$ correspond to partial differentiations with respect to the only variable z_3, while $\mathbf{grad}_{y_\alpha}^\mathbf{s}$ and $\mathbf{div}_{y_\alpha}$ are the differential operators with regard to the two microscopic variables y_α.
As a second step, it is necessary to presuppose the order of magnitude of the loadings which are applied to the structure. Especially, we pose :

$$\left\{\begin{array}{l} \bar{\sigma}_{33}^\varepsilon(x_1, x_2) = \varepsilon^1.\bar{\sigma}_{33}(y_1, y_2) \\ \bar{\sigma}_{\alpha 3}^\varepsilon(x_1, x_2) = \varepsilon^2.\bar{\sigma}_{\alpha 3}(y_1, y_2) \end{array}\right. \tag{4}$$

Furthermore, the elasticity moduli a_{ijkl}^ε are assumed to be independent of ε, so we have :

$$\mathbf{a}^\varepsilon(x_\alpha) = \mathbf{a}(y_\alpha) \tag{5}$$

Third, following a standard technique, the solution $\mathbf{u}^\varepsilon$ of the (P^ε) problem is sought in the following form[6] :

$$\mathbf{u}^{(I)}(\mathbf{x}) = \hat{u}_\alpha^0(z_3)\mathbf{e}_\alpha + \varepsilon\mathbf{u}^1(z_3, y_\alpha) + \varepsilon^2\mathbf{u}^2(z_3, y_\alpha) + ... \tag{6}$$

In (6), the superscript (I) rather than ε is used in order to distinguish throughout this paper the *interior* solution (I), which satisfy the equations $(1)_{1-4}$ but not the boundary conditions $(1)_{5-6}$, from the exact 3D solution ($\mathbf{u}^\varepsilon, \boldsymbol{\sigma}^\varepsilon$), which verify all the equations (1).

When introducing the relations (2)-(6) into the P^ε problem (1) and equating the terms of a same order with respect to ε, we replace the problem P^ε by a family of problems. The fields involved in the latter are functions of the two kinds of variables y_α and z_3, but no longer depend on the small parameter ε. So, when treating the z_3- and y_α-coordinates as independent, and considering the fields function of the only variable z_3 as given data, we can regard each of these problems as a *microscopic* 2D-problem, which is posed on the scaled section S. In that sense, these successive problems will be denoted herein P_{2D}^k, where the superscript k stands for the order of the current problem with respect to ε.

As it will be seen in the next subsection, the solution of these problems enables us to determine the microscopic parts of the expansion (6). Then, expressing the existence conditions of solutions for the P_{2D}^k problems, we obtain the formulation of *macroscopic* 1D-problems, denoted by P_{1D}^k, the solution of which gives the macroscopic parts (function of z_3) of the field (6).

The set of microscopic 2D-problems

As explained previously, the formulation of an infinite set of microscopic P_{2D}^k problems is derived, with k starting from -1. All these problems are posed on the scaled cross-section S, deduced from S^ε through the scaling (2).

The first problem P_{2D}^{-1} occurs for $k = -1$ and can be written as follows :

$$\left\{\begin{array}{l} \mathbf{div}_{y_\alpha}\boldsymbol{\sigma}^0 = \mathbf{0} \\ \boldsymbol{\sigma}^0 = \mathbf{a}(y_\alpha) : \mathbf{e}^0 \\ \mathbf{e}^0 = \mathbf{grad}^{\mathbf{s}}_{y_\alpha}(\mathbf{u}^1) + \mathbf{grad}^{\mathbf{s}}_{z_3}(\mathbf{u}^0) \\ \boldsymbol{\sigma}^0.\mathbf{n} = \mathbf{0} \quad on \quad \partial S \end{array}\right. \tag{7}$$

The only data of the problem are thus contained in the tensor $\mathbf{grad}^{\mathbf{s}}_{z_3}(\mathbf{u}^0)$. According to the form of the field $\mathbf{u}^0$ given in (6), it can easily be established that this problem possesses a *direct* solution which is :

$$\left\{\begin{array}{l} \mathbf{u}^1_{part} = -y_\alpha.\partial_3\hat{u}^0_\alpha(z_3).\mathbf{e_3} \\ \boldsymbol{\sigma}^0 = \mathbf{e}^0 = \mathbf{0} \end{array}\right. \tag{8}$$

In that sense, the two data $\partial_3\hat{u}^0_\alpha(z_3)$ contained in the term $\mathbf{grad}^{\mathbf{s}}_{z_3}(\mathbf{u}^0)$ do not constitute *effective* data, since the associated solution corresponds to a zero deformation state[6] .
The displacement field given in (8) is obtained up to an element of $\Re$, defined as :

$$\Re = \{\mathbf{v}(z_3, y_\alpha)/\mathbf{v} = \hat{v}_i(z_3)\mathbf{e_i} + \varphi(z_3)[y_1\mathbf{e_2} - y_2\mathbf{e_1}]\} \tag{9}$$

The elements of $\Re$ are such that the deformations $\mathbf{grad}^{\mathbf{s}}_{y_\alpha}(\mathbf{v})$ are equal to zero, and $\Re$ constitutes thus the set of the trial solutions of the microscopic P_{2D}^k problems. Consequently, the complete solution of (7) has to be written as follows :

$$\left\{\begin{array}{l} \mathbf{u}^1 = \hat{u}^1_i(z_3).\mathbf{e_i} + \varphi^1(z_3)[y_1.\mathbf{e_2} - y_2.\mathbf{e_1}] \\ \qquad -y_\alpha.\partial_3\hat{u}^0_\alpha(z_3).\mathbf{e_3} \\ \qquad \equiv \tilde{\mathbf{u}}^1(z_3, y_\alpha) \end{array}\right. \tag{10}$$

Since $\boldsymbol{\sigma}^0 = 0$, the second microscopic problem P_{2D}^0 consists in finding the fields $\boldsymbol{\sigma}^1$, $\mathbf{e}^1$ and $\mathbf{u}^2$, solutions of :

$$\left\{\begin{array}{l} \mathbf{div}_{y_\alpha}\boldsymbol{\sigma}^1 = \mathbf{0} \\ \boldsymbol{\sigma}^1 = \mathbf{a}(y_\alpha) : \mathbf{e}^1 \\ \mathbf{e}^1 = \mathbf{grad}^{\mathbf{s}}_{y_\alpha}(\mathbf{u}^2) + \mathbf{grad}^{\mathbf{s}}_{z_3}(\mathbf{u}^1) \\ \boldsymbol{\sigma}^1.\mathbf{n} = \mathbf{0} \quad on \quad \partial S \end{array}\right. \tag{11}$$

According to the expression (10) of $\mathbf{u}^1$ obtained at the preceding order, the macroscopic data contained in $\mathbf{grad}^{\mathbf{s}}_{z_3}(\mathbf{u}^1)$ are found to be $\partial_3\hat{u}^1_\alpha(z_3)$, $\partial_3\hat{u}^1_3(z_3)$, $\partial_{33}\hat{u}^0_\alpha(z_3)$, $\partial_3\varphi^1(z_3)$. The two data $\partial_3\hat{u}^1_\alpha(z_3)$ will provide a direct solution $\mathbf{u}^2_{part}$ similar to expression (8). The four other data correspond respectively to a macroscopic extension, two macroscopic curvatures and a macroscopic torsion rotation. The problem (11) does not have an analytical form solution in general, excepted in the case of homogeneous isotropic rods, and of some particular cases of heterogeneous cross-sections[5] .
Due to the linearity of (11), the displacement field $\mathbf{u}^2$ can be expressed as a linear function of these four *effective* data. Adding the direct solution $\mathbf{u}^2_{part}$ provided by the two data $\partial_3\hat{u}^1_\alpha(z_3)$ as well as the rigid motion of $\Re$, the complete displacement field at the second order recovers the following form :

$$\left\{\begin{array}{l} \mathbf{u}^2 = \tilde{\mathbf{u}}^2 + \boldsymbol{\chi}^{1E}(y_\beta).\partial_3\hat{u}^1_3(z_3) \\ \qquad +\boldsymbol{\chi}^{1C_\alpha}(y_\beta).\partial_{33}\hat{u}^0_\alpha(z_3) + \boldsymbol{\chi}^{1T}(y_\beta).\partial_3\varphi^1(z_3) \end{array}\right. \tag{12}$$

where :

$$\left\{\begin{array}{l} \tilde{\mathbf{u}}^2(z_3, y_\beta) = \hat{u}^2_i(z_3).\mathbf{e_i} + \varphi^2(z_3)[y_1.\mathbf{e_2} - y_2.\mathbf{e_1}] \\ \qquad -y_\alpha.\partial_3\hat{u}^1_\alpha(z_3).\mathbf{e_3} \end{array}\right. \tag{13}$$

For later consistency of notations, we introduce the four-components vector $\tilde{\mathbf{e}}^1(z_3)$ and the 3×4 matrix $\boldsymbol{\chi}^1(y_\beta)$ so that we have :

$$\mathbf{u}^2 = \tilde{\mathbf{u}}^2(z_3, y_\beta) + \boldsymbol{\chi}^1(y_\beta).\tilde{\mathbf{e}}^1(z_3) \tag{14}$$

with :

$$\left\{\begin{array}{l} \tilde{\mathbf{e}}^1(z_3) = {}^t\{\partial_3\hat{u}^1_3, \partial_{33}\hat{u}^0_1, \partial_{33}\hat{u}^0_2, \partial_3\varphi^1\} \\ \boldsymbol{\chi}^1(y_\beta) = [\boldsymbol{\chi}^{1E}, \boldsymbol{\chi}^{1C_1}, \boldsymbol{\chi}^{1C_2}, \boldsymbol{\chi}^{1T}] \end{array}\right. \tag{15}$$

In (15), the four effective data have been grouped in the vector $\tilde{\mathbf{e}}^1(z_3)$, with the result that the latter represents the first-order macroscopic strain vector.
In the same manner as the displacement field, the stress field $\boldsymbol{\sigma}^1$ solution of P_{2D}^0 has a linear expression with regard to the data :

$$\left\{\begin{array}{l} \boldsymbol{\sigma}^1 = \boldsymbol{\tau}^{1E}(y_\beta).\partial_3\hat{u}^1_3(z_3) + \boldsymbol{\tau}^{1C_\alpha}(y_\beta).\partial_{33}\hat{u}^0_\alpha(z_3) \\ \qquad +\boldsymbol{\tau}^{1T}(y_\beta).\partial_3\varphi^1(z_3) \end{array}\right. \tag{16}$$

with

$$\left\{\begin{array}{l} \tau^{1E}_{ij} = a_{ij33} + a_{ijk\beta}\partial_{y_\beta}\chi^{1E}_k \\ \tau^{1C_\alpha}_{ij} = -y_\alpha a_{ij33} + a_{ijk\beta}\partial_{y_\beta}\chi^{1C_\alpha}_k \\ \tau^{1T}_{ij} = -y_2 a_{ij13} + y_1 a_{ij23} + a_{ijk\beta}\partial_{y_\beta}\chi^{1T}_k \end{array}\right. \tag{17}$$

which will be formally denoted as :

$$\boldsymbol{\sigma}^1 = \boldsymbol{\tau}^1(y_\beta).\tilde{\mathbf{e}}^1(z_3) \tag{18}$$

where $\boldsymbol{\tau}^1(y_\beta)$ corresponds to the regroupement of the four elementary stress tensors $\boldsymbol{\tau}^{1E}, \boldsymbol{\tau}^{1C_\alpha}, \boldsymbol{\tau}^{1T}$.

The formulation as well as the expression of the solution of the first two microscopic problems have been so far given. Treating exactly in a same manner the higher-order problems $P_{2D}^k, k \geq 1$, a generalization of the latter results can be easily outlined.

For an arbitrary power k of the small parameter ε, the P_{2D}^k problem posed on the scaled section S consists in finding the fields $\boldsymbol{\sigma}^{k+1}$, $\mathbf{e}^{k+1}$ and $\mathbf{u}^{k+2}$ satisfying the following equations :

$$\left\{\begin{array}{l} \mathbf{div}_{y_\alpha}\boldsymbol{\sigma}^{k+1} = -\mathbf{div}_{z_3}\boldsymbol{\sigma}^k \\ \boldsymbol{\sigma}^{k+1} = \mathbf{a}(y_\alpha) : \mathbf{e}^{k+1} \\ \mathbf{e}^{k+1} = \mathbf{grad}^{\mathbf{s}}_{y_\alpha}(\mathbf{u}^{k+2}) + \mathbf{grad}^{\mathbf{s}}_{z_3}(\mathbf{u}^{k+1}) \\ \boldsymbol{\sigma}^{k+1}.\mathbf{n} = \mathbf{0} \quad on \quad \partial S \end{array}\right. \tag{19}$$

with $k \geq -1$ and where the negative powers of $\boldsymbol{\sigma}^k$ and $\mathbf{e}^k$ vanish.

When solving this problem P_{2D}^k at order k, we consider that the preceding P_{2D}^{k-1} problem has already been solved and thus that the fields $\boldsymbol{\sigma}^k$ and $\mathbf{u}^{k+1}$ have been determined. Consequently, the terms $\mathbf{div}_{z_3}\boldsymbol{\sigma}^k$ and $\mathbf{grad}^{\mathbf{s}}_{z_3}(\mathbf{u}^{k+1})$ constitute macroscopic *given* fields for the current problem P_{2D}^k : the first one can be regarded as a fictive volume force and the second one as an initial strain state in the cross-section S.

The variational form of problem (19) consists in finding the field $\mathbf{u}^{k+2}$ in a space $W(S)$ of sufficiently regular functions $\boldsymbol{\psi}$ such that :

$$\left\{\begin{array}{l} \forall \quad \boldsymbol{\psi} \in W(S), \\ \int_S \boldsymbol{\sigma}^{k+1} : \mathbf{grad}^{\mathbf{s}}_{y_\alpha}(\boldsymbol{\psi})\, dS = \int_S \mathbf{div}_{z_3}\boldsymbol{\sigma}^k.\boldsymbol{\psi}\, dS \end{array}\right. \tag{20}$$

where the stress field $\boldsymbol{\sigma}^{k+1}$ is related to the field $\mathbf{u}^{k+2}$ following $(19)_2$.

According to the variational form (20), it can be established that the existence of a solution for the P_{2D}^k problem is guaranteed provided that the data $\mathbf{div}_{z_3}\boldsymbol{\sigma}^k$ verify the following relation :

$$\forall \quad \mathbf{v} \in \Re, \quad \int_S \mathbf{div}_{z_3}\boldsymbol{\sigma}^k.\mathbf{v}\, dS = 0 \tag{21}$$

The compatibility condition (21) will enable us to formulate the *macroscopic* problems, as we shall see in the next section.

Under this necessary condition (21), the solutions $\boldsymbol{\sigma}^{k+1}$, $\mathbf{e}^{k+1}$ and $\mathbf{u}^{k+2}$ (determined up to an element of $\Re$) exist and can be linearly expressed with respect to the data contained in $\mathbf{div}_{z_3}\boldsymbol{\sigma}^k$ and $\mathbf{grad}^{\mathbf{s}}_{z_3}(\mathbf{u}^{k+1})$.

When solving the P_{2D}^0 problem (11), it has been shown that the data $\mathbf{grad}^{\mathbf{s}}_{z_3}(\mathbf{u}^1)$ naturally introduces the first order macroscopic strain $\tilde{\mathbf{e}}^1(z_3)$, defined in $(15)_1$. In a same way, the data $\mathbf{div}_{z_3}\boldsymbol{\sigma}^k$ and $\mathbf{grad}^{\mathbf{s}}_{z_3}(\mathbf{u}^{k+1})$ of the P_{2D}^k problem are found to involve the contribution of the $(k+1)$th-order macroscopic strain $\tilde{\mathbf{e}}^{k+1}(z_3)$, but also the one of the first gradient of the kth-order strain (i.e. $\partial_3\tilde{\mathbf{e}}^k(z_3)$), the second gradient of the $(k-1)$th-order strain (i.e. $\partial_{33}\tilde{\mathbf{e}}^{k-1}(z_3)$), and so on until the kth gradient of the first-order strain, $\partial_3^k\tilde{\mathbf{e}}^1(z_3)$. Consequently, assuming that the compatibility condition (21) is satisfied, the solutions $\mathbf{u}^{k+2}$ and $\boldsymbol{\sigma}^{k+1}$ of (19) can be formally written as follows :

$$\left\{\begin{array}{l} \mathbf{u}^{k+2} = \tilde{\mathbf{u}}^{k+2}(z_3, y_\beta) + \boldsymbol{\chi}^1(y_\beta).\tilde{\mathbf{e}}^{k+1}(z_3) \\ \quad +\boldsymbol{\chi}^2(y_\beta).\partial_3\tilde{\mathbf{e}}^k(z_3) + \ldots + \boldsymbol{\chi}^{k-1}(y_\beta).\partial_3^k\tilde{\mathbf{e}}^1(z_3) \\ \boldsymbol{\sigma}^{k+1} = \boldsymbol{\tau}^1(y_\beta).\tilde{\mathbf{e}}^{k+1}(z_3) + \boldsymbol{\tau}^2(y_\beta).\partial_3\tilde{\mathbf{e}}^k(z_3) \\ \quad +\ldots + \boldsymbol{\tau}^{k+1}(y_\beta).\partial_3^k\tilde{\mathbf{e}}^1(z_3) \end{array}\right. \tag{22}$$

with :

$$\left\{\begin{array}{l} \tilde{\mathbf{e}}^k(z_3) = {}^t\{\partial_3\hat{u}_3^k, \partial_{33}\hat{u}_1^{k-1}, \partial_{33}\hat{u}_2^{k-1}, \partial_3\varphi^k\} \\ \boldsymbol{\chi}^k(y_\beta) = [\boldsymbol{\chi}^{kE}, \boldsymbol{\chi}^{kC_1}, \boldsymbol{\chi}^{kC_2}, \boldsymbol{\chi}^{kT}] \end{array}\right. \tag{23}$$

with $k \geq -1$ and where quantities $\tilde{\mathbf{e}}^k$ vanish for $k \leq 0$.

Therefore, solving in sequence the 2D-problems P_{2D}^k enables to derive the following expression of the asymptotic outer solution $\mathbf{u}^{(I)}$ (6) :

$$\left\{\begin{array}{l} \mathbf{u}^{(I)}(\mathbf{x}) = \hat{u}_\alpha^0(z_3)\mathbf{e}_\alpha + \varepsilon^p\{\tilde{\mathbf{u}}^p(z_3, y_\beta) \\ \qquad +\boldsymbol{\chi}^1(y_\beta)\tilde{\mathbf{e}}^{p-1}(z_3) \\ \qquad +\ldots + \boldsymbol{\chi}^{p-1}(y_\beta)\partial_3^{p-2}\tilde{\mathbf{e}}^1(z_3)\} \\ \boldsymbol{\sigma}^{(I)}(\mathbf{x}) = \varepsilon^p\{\boldsymbol{\tau}^1(y_\beta)\tilde{\mathbf{e}}^p(z_3) \\ \qquad +\ldots + \boldsymbol{\tau}^p(y_\beta)\partial_3^{p-1}\tilde{\mathbf{e}}^1(z_3)\} \end{array}\right. \tag{24}$$

In (24), the displacement fields $\boldsymbol{\chi}^p(y_\beta)$ and the stress fields $\boldsymbol{\tau}^p(y_\beta)$ constitute the microscopic parts of the outer expansions, and we have seen throughout this section how to determine these fields from the solution of 2D-problems posed on the cross-section. On the other hand, the fields $\tilde{\mathbf{u}}^p(z_3, y_\beta)$ and their respective gradients $\tilde{\mathbf{e}}^p(z_3)$ characterize the macroscopic parts of the outer state $(\mathbf{u}^{(I)}, \boldsymbol{\sigma}^{(I)})$, and have now to be found, in order to completely define the outer solution. The way of obtaining the latter fields will be presented in the next section.

The set of macroscopic 1D-problems

As already mentioned, the equilibrium equations corresponding to the unknown displacement fields $\tilde{\mathbf{u}}^p$ are obtained from the compatibility condition (21). Expressing this condition for the 2D-problems P_{2D}^k and P_{2D}^{k+1} leads indeed to the formulation of the macroscopic 1D-problems P_{1D}^k.

Governing equations of the first 1D-problem

Expressing the relation (21) with the four 'elementary' functions of $\Re$: $\hat{v}_3(z_3)\mathbf{e_3}$, $\hat{v}_\alpha(z_3)\mathbf{e}_\alpha$ and $(y_1\mathbf{e_2} - y_2\mathbf{e_1})$ on one hand, and putting $\boldsymbol{\psi} = y_\alpha\mathbf{e_3}$

in (20) on the other hand, lead for $k = 1$ to :

$$\begin{cases} \partial_3 N^1 = 0 \\ \partial_3 T^1_\alpha = 0 \\ \partial_3 M^1_3 = 0 \\ -T^2_\alpha + \partial_3 M^1_\alpha = 0 \end{cases} \tag{25}$$

where the kth-order beam stresses $N^k(z_3)$, $T^k_\alpha(z_3), M^k_\alpha(z_3), M^k_3(z_3)$ correspond respectively to the axial force, the transverse shearing forces, the bending moments and the twisting moment, defined by :

$$\begin{cases} N^k(z_3) = \int_S \sigma^k_{33}\, dS \\ T^k_\alpha(z_3) = \int_S \sigma^k_{\alpha 3}\, dS \\ M^k_\alpha(z_3) = \int_S -y_\alpha \sigma^k_{33}\, dS \\ M^k_3(z_3) = \int_S (y_1 \sigma^k_{23} - y_2 \sigma^k_{13})\, dS \end{cases} \tag{26}$$

In the same way, one obtains from the P^2_{2D} problem the relation $\partial_3 T^2_\alpha = 0$.
Furthermore, it can be proved that the first-order shearing forces T^1_α are zero, so that equations $(25)_2$ are identically satisfied. Consequently, the equilibrium equations of the P^1_{1D} problem recover the classical form :

$$\begin{cases} \partial_3 N^1 = 0 \\ \partial_{33} M^1_\alpha = 0 \\ \partial_3 M^1_3 = 0 \end{cases} \tag{27}$$

The constitutive relations of the P^1_{1D} problem can be defined as :

$$\begin{cases} \tilde{\boldsymbol{\sigma}}^1 = \mathcal{A}^1.\tilde{\mathbf{e}}^1 \\ \textit{with} \quad \tilde{\boldsymbol{\sigma}}^1(z_3) = {}^t\{N^1, M^1_1, M^1_2, M^1_3\} \end{cases} \tag{28}$$

The components of the 4×4 matrix $\mathcal{A}^1$ are computed from the quantities τ^1_{i3} defined in (17). Thus, $\mathcal{A}^1_{11}$ is the stretching stiffness, $\mathcal{A}^1_{22}$ and $\mathcal{A}^1_{33}$ the two bending stiffnesses, $\mathcal{A}^1_{44}$ the twisting stiffness and the extra-diagonal quantities are the different coupling terms. Note that the effective stiffness matrix $\mathcal{A}^1$ is determined from the solution of the first-order 2D-problem P^1_{2D}. It can be proved that $\mathcal{A}^1$ is symmetric and positively definite[6] .

Governing equations of the kth 1D-problem

Applying recursively the method used for the derivation of equilibrium equations for the P^1_{1D} problem, one obtains, for the P^k_{1D} problem, $k \geq 2$:

$$\begin{cases} \partial_3 N^k = 0 \\ \partial_{33} M^k_\alpha = 0 \\ \partial_3 M^k_3 = 0 \end{cases} \tag{29}$$

From the solution $\boldsymbol{\sigma}^{k+1}$ of the P^k_{2D} problem, it can be seen that the kth-order constitutive relations take the following form :

$$\begin{cases} \tilde{\boldsymbol{\sigma}}^k = \mathcal{A}^1.\tilde{\mathbf{e}}^k + \mathcal{A}^2.\partial_3 \tilde{\mathbf{e}}^{k-1} \\ \qquad +\mathcal{A}^3.\partial_{33}\tilde{\mathbf{e}}^{k-2} + ... + \mathcal{A}^k.\partial_3^{k-1}\tilde{\mathbf{e}}^1 \end{cases} \tag{30}$$

with $\tilde{\boldsymbol{\sigma}}^k(z_3) = {}^t\{N^k, M^k_1, M^k_2, M^k_3\}$. The 4×4 matrix $\mathcal{A}^p$, $p \geq 2$, is obtained from the stress fields $\boldsymbol{\tau}^p$, introduced in (24), which corresponds to the solution of the P^{p-1}_{2D} problem, considering as only data $\partial_3^{p-1}\tilde{\mathbf{e}}^1$.

Contrary to the first order effective stiffness matrix $\mathcal{A}^1$, the higher-order stress-strain matrices $\mathcal{A}^k$, $k \geq 2$, are not necessarily symmetric or positively definite tensors. Especially, the second order one, $\mathcal{A}^2$, appears to be antisymmetric and even equal to zero following certain symmetry properties of the cross-section. A study of these symmetry properties can be found in[7] in the case of 3D periodic media.

It must be noted that the P^k_{1D} problem is treated once the lower-order 1D-problems are solved. Therefore, the strains $\tilde{\mathbf{e}}^1, ..., \tilde{\mathbf{e}}^{k-1}$ of relation (30) are known, so that the effective unknown strain of that problem is $\tilde{\mathbf{e}}^k$, associated to the unknown displacement field $\{\hat{u}^{k-1}_\alpha, \hat{u}^k_3, \varphi^k\}$. As a matter of fact, the higher-order effects may be seen as fictive volume loadings, as pointed out in[8] .

In order to solve the macroscopic 1D-problem P^k_{1D}, it remains to complete the governing equations (29)-(30) with the boundary conditions at the ends $z_3 = 0$ and $z_3 = L$. This will be the subject of the next section.

Corrections due to end effects

Preliminary remarks

The successive terms of the outer expansion $(\mathbf{u}^{(I)}, \boldsymbol{\sigma}^{(I)})$ are not able to satisfy arbitrarily prescribed end sections conditions, and a specific study appears to be necessary.

One way to take into account the edge effects is to complement the outer expansion (6) by an inner expansion which gives the correct beam behavior near the end sections. However, such an approach requires to solve boundary layer problems at each order. A more *direct* method enables to find the appropriate boundary conditions for the outer solution. This method, so called the *decay analysis technique*, is based on Maxwell-Betti's theorem and provides the set of boundary conditions for the outer expansion, involving the solution of canonical problems, which can be solved once for all.

This method has been initially proposed for plate problems[4] , and has been applied to the cases of homogeneous beams[9] , and of layered orthotropic beams in a plane-stress analysis[2] . The main steps of this method are presented in the next section.

The decay analysis method

The aim of the decay analysis technique proposed in[4] is to derive the correct boundary conditions which have to be prescribed on the interior asymptotic expansions (24) in order that the difference between the latter and the exact 3D solution remains significant only near the edges.

For example, when considering the boundary conditions on the end-section Γ_0^ε ($x_3 = 0$), the problem is to find the boundary conditions for the interior solution such that one has :

$$(\boldsymbol{\sigma}^{(I)}, \mathbf{u}^{(I)}) - (\boldsymbol{\sigma}^{\varepsilon}, \mathbf{u}^{\varepsilon}) \to 0 \qquad as \quad x_3 \dashrightarrow \infty \quad (31)$$

We recall that, in (31), the superscript (I) stands for the *interior* solution while $(\boldsymbol{\sigma}^{\varepsilon}, \mathbf{u}^{\varepsilon})$ corresponds to the exact 3D elastostatic state which verifies (1).
To obtain the boundary conditions such that the decaying condition (31) holds, the method proposed in[4] relies on the use of the reciprocity theorem (Maxwell-Betti's theorem), as explained below.
Let $(\boldsymbol{\sigma}^{(1)}, \mathbf{u}^{(1)})$ and $(\boldsymbol{\sigma}^{(2)}, \mathbf{u}^{(2)})$ be two elastostatic states which verify the equations $(1)_{1-4}$, i.e. the equilibrium equations, compatibility relations as well as the traction free condition of the 3D problem.
Applying the reciprocity theorem with these two states leads then to the relation :

$$\int_{\Gamma_0^\varepsilon \cup \Gamma_L^\varepsilon} [\sigma_{ij}^{(1)} u_i^{(2)} - \sigma_{ij}^{(2)} u_i^{(1)}] n_j \, dS^\varepsilon = 0 \qquad (32)$$

The main idea of the method amounts then to choosing for one of the states (1) or (2) the difference between the exact solution of the P^ε problem and the interior solution. For example, let state (1) be that difference, so that :

$$(\boldsymbol{\sigma}^{(1)}, \mathbf{u}^{(1)}) = (\boldsymbol{\sigma}^{(I)}, \mathbf{u}^{(I)}) - (\boldsymbol{\sigma}^{\varepsilon}, \mathbf{u}^{\varepsilon}) \qquad (33)$$

If we now enforce the difference $(\boldsymbol{\sigma}^{(1)}, \mathbf{u}^{(1)})$ to be rapidly decaying when going away from the end-section Γ_0^ε, it comes from the decay condition (31) that the integral on the edge Γ_L^ε vanishes so that (32) is reduced to :

$$\int_{\Gamma_0^\varepsilon} [\sigma_{ij}^{(1)} u_i^{(2)} - \sigma_{ij}^{(2)} u_i^{(1)}] n_j \, dS = 0 \qquad (34)$$

Since (34) has to be verified for any regular state $(\boldsymbol{\sigma}^{(2)}, \mathbf{u}^{(2)})$, it follows that the relation (34) will provide the boundary conditions on the interior fields $(\boldsymbol{\sigma}^{(I)}, \mathbf{u}^{(I)})$ so that the edge effects decay rapidly, as we shall see in the following.

Let us now exploit the equation (34) to derive the six expected boundary conditions for the outer expansions $(\boldsymbol{\sigma}^{(I)}, \mathbf{u}^{(I)})$ at each end-section.

The end section $z_3 = 0$

In the case of the clamped end-section, the part σ_{i3}^ε involved in $\boldsymbol{\sigma}^{(1)}$ is unknown. Consequently, the state (2) has to be selected such that $\mathbf{u}^2 = \mathbf{0}$. Such states (2) are given by the six fundamental solutions of the cantilevered beam, submitted to an unit tension $^{(BE)}$, bending $^{(BB_\alpha)}$, flexure $^{(BF_\alpha)}$ and torsion $^{(BT)}$. So, from (34), the following boundary conditions are derived for the outer expansion :

$$\begin{aligned} &\int_{\Gamma_0} \sigma_{i3}^{(2)} u_i^{(I)} \, dS = 0 \\ &with \quad .^{(2)} \equiv .^{(BE)}, .^{(BB_\alpha)}, .^{(BF_\alpha)}, .^{(BT)} \end{aligned} \qquad (35)$$

where Γ_0 is the scaled end cross-section.

The end section $z_3 = L$

Since we have stress edge-data, the states (2) are given by the six rigid body displacements. Inserting these states in (34) and taking into account the assumptions (4), one obtains :

$$\begin{cases} N^{(I)}(L) = \varepsilon \int_{\Gamma_L} \bar{\sigma}_{33} \, dS \\ T_\alpha^{(I)}(L) = \varepsilon^2 \int_{\Gamma_L} \bar{\sigma}_{\alpha 3} \, dS \\ M_\alpha^{(I)}(L) = \varepsilon \int_{\Gamma_L} -y_\alpha \bar{\sigma}_{33} \, dS \\ M_3^{(I)}(L) = \varepsilon^2 \int_{\Gamma_L} (y_1 \bar{\sigma}_{23} - y_2 \bar{\sigma}_{13}) \, dS \end{cases} \qquad (36)$$

So in that case, a justification of the Saint-Venant principle is obtained, since the boundary conditions involve load resultants.

The macroscopic outer solution

Inserting (24) in (35) and (36), and using the overall beam equilibrium equations involving the solution of the canonical problems, one can derive the boundary conditions at each order and then solve successively the macroscopic 1D-problems.

Considering the first 1D-problem P_{1D}^1, we are led to the following boundary conditions :

$$\begin{cases} \hat{u}^0_\alpha(0) = \partial_3\hat{u}^0_\alpha(0) = \hat{u}^1_3(0) = \varphi^1(0) = 0 \\ N^1(L) = \int_{\Gamma_L} \bar{\sigma}_{33}\, dS; \quad T^2_\alpha(L) = \int_{\Gamma_L} \bar{\sigma}_{\alpha 3}\, dS \\ M^1_\alpha(L) = \int_{\Gamma_L} -y_\alpha \bar{\sigma}_{33}\, dS; \quad M^1_3(L) = 0 \end{cases} \tag{37}$$

which are the classical engineering boundary conditions in the Euler-Bernoulli theory. Consequently, the method used here enables to rigorously justify this theory (governing equations *and* boundary conditions) as the leading term of the outer expansion.

Let us focus now on the 1D-problem involving the full macroscopic outer expansion.
Solving the first 1D-problem P^1_{1D}, from (27), (28) and the positive-definiteness of $\mathcal{A}^1$, it turns out that $\partial_{33}\tilde{\mathbf{e}}^1(z_3) = \mathbf{0}$. As a result, the second 1D-problem P^2_{1D} leads to $\partial_{33}\tilde{\mathbf{e}}^2(z_3) = \mathbf{0}$, and in a recursive manner, one can prove that the differential equations of the pth-order macroscopic problem are of the form :

$$\begin{cases} \mathcal{A}^{homl}_{11}\partial^2_3\hat{u}^p_3 + \mathcal{A}^{homl}_{12}\partial^3_3\hat{u}^{p-1}_1 + \mathcal{A}^{homl}_{13}\partial^3_3\hat{u}^{p-1}_2 \\ \qquad +\mathcal{A}^{homl}_{14}\partial^2_3\varphi^p = 0 \\ \mathcal{A}^{homl}_{j1}\partial^3_3\hat{u}^p_3 + \mathcal{A}^{homl}_{j2}\partial^4_3\hat{u}^{p-1}_1 + \mathcal{A}^{homl}_{j3}\partial^4_3\hat{u}^{p-1}_2 \\ \qquad +\mathcal{A}^{homl}_{j4}\partial^3_3\varphi^p = 0, \qquad j = 2,3 \\ \mathcal{A}^{homl}_{41}\partial^2_3\hat{u}^p_3 + \mathcal{A}^{homl}_{42}\partial^3_3\hat{u}^{p-1}_1 + \mathcal{A}^{homl}_{43}\partial^3_3\hat{u}^{p-1}_2 \\ \qquad +\mathcal{A}^{homl}_{44}\partial^2_3\varphi^p = 0 \end{cases} \tag{38}$$

This remarkable result enables to define the problem involving the macroscopic part of the full outer expansion, defined by :

$$\begin{cases} \hat{u}_\alpha(z_3) = \hat{u}^0_\alpha(z_3) + \varepsilon\hat{u}^1_\alpha(z_3) + \ldots \\ \hat{u}_3(z_3) = \hat{u}^1_3(z_3) + \varepsilon\hat{u}^2_3(z_3) + \ldots \\ \varphi(z_3) = \varphi^1(z_3) + \varepsilon\varphi^2(z_3) + \ldots \end{cases} \tag{39}$$

The differential equations of this problem are obtained from the addition of (38) at each order : it results that the macroscopic part of the full outer expansion verifies the differential equations of the generalized Euler-Bernoulli's theory. Lastly, the expression of the boundary conditions at the end-sections Γ_0, Γ_L are directly given by (35), (36).

Therefore, it is possible to obtain the macroscopic part of the outer solution from the solution of a single 1D-problem, without determining any of the terms of the expansion. Such a property results from the assumption that the beam is loaded only on its end-sections, otherwise, a higher-order gradient beam model is obtained. A similar result can be found in the case of homogeneous isotropic infinite plate strip, free of external loads in the interior and free of surface tractions on its upper and lower faces[4] .

Numerical example

To illustrate the method, a sandwich beam with a square cross-section is considered. The corresponding geometric and material properties are shown in Fig. 2. The beam length is 300 mm (so that $\varepsilon = 1/15$) and the beam is cantilevered at the end Γ^ε_0, and subjected to a transverse shear force at the end Γ^ε_L in the $\mathbf{e}_1$ direction.

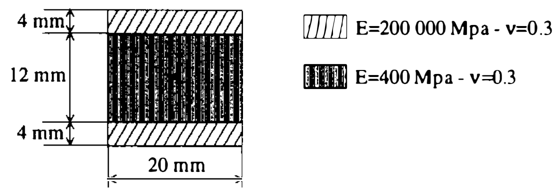

Fig. 2 beam cross-section geometry and material

The exact 3D solution is computed with a detailed finite element model of the structure, obtained from an extrusion of the cross-section model used to solve the microscopic problems.

On the other hand, the asymptotic method is applied. In a first step, the microscopic 2D-problems are treated. Given that the kth-order strain gradient, with $k \geq 2$, are zero, the computations only concern the solution of the first two problems P^1_{2D} and P^2_{2D}.
Moreover, the 3D numerical solution of the canonical problems is computed considering a beam of a finite length. This length is taken sufficiently large so that the way of loading does not influence the stress distribution at the fixed end. The boundary conditions for the 1D-problem corresponding to the macroscopic part of the full outer expansion can then be calculated.
Finally, the analytical solution of this problem is derived, enabling to determine the full asymptotic macroscopic deflection $\hat{u}^0_1 + \varepsilon\hat{u}^1_1 + \varepsilon^2\hat{u}^2_1 + \varepsilon^3\hat{u}^3_1$, which is found to stop at the third order.
The latter is compared to the deflection of the centroidal axis given by the 3D computation, see Fig. 3, where the contribution of the successive terms of the outer expansion is shown. It is obvious that the zero-th-order solution $\hat{u}^0_1$ (which is the solution of the P^1_{1D} problem), gives a very bad estimation of the exact solution, emphasizing the fact that the beam studied here is weak in shear.

The term $\hat{u}_1^1$ is found to be zero, and a significant improvement is given by the second-order solution (which is of the same order of magnitude of the zero-th order one), and the agreement between the solution computed up to the third-order (i.e. the full outer solution) and the 3D solution is very good. Moreover, it can be noticed that the third-order approximation differs from the second-order one only through a non-zero deflection at the built-in end, underlining the contribution of edge effects.

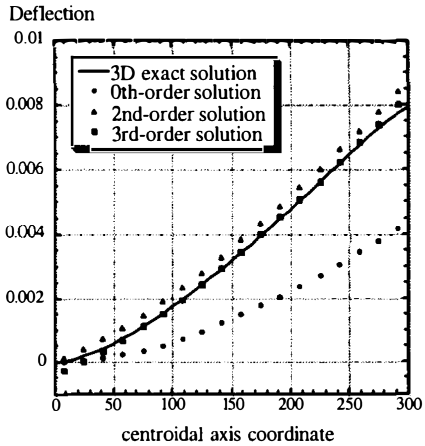

Fig. 3 Deflection of the cantilever beam

Furthermore, far from the ends, and using the solution of the microscopic 2D-problems and expression (24), it is possible to calculate the distributions of displacements, strains, and stresses in the cross-section, which are found to be very closed to those obtained from the 3D computations.

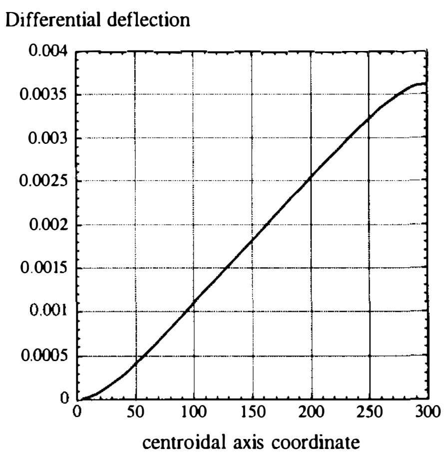

Fig. 4 Difference between 3D deflection and 0th-order deflection

It is also interesting to study the results which could be given from a Timoshenko's beam model. Following this theory, the overall beam deflection is obtained adding a linear term with respect to the centroidal axis coordinate to the zero-th-order deflection. But in point of fact, the difference between the 3D exact solution and the zero-th-order solution, shown in Fig.4, is found to be non-linear because of edge effects. Thus, whatever the value of the shear coefficient, it can be seen that the Timoshenko's model will lead to a solution which cannot fit the 3D solution, even far from the edges.

Concluding Remarks

In this paper, the formal asymptotic expansion method is applied to derive of a 1D-model for an elastic beam. In a preliminary step, this method enables a rigorous accounting for non-classical effects, without any assumption of their relative order of magnitude, through the solution of successive microscopic 2D-problems. Then, considering the overall beam response, it has been proved that, in the case where the beam is loaded only at its end-sections, the full outer expansion is the solution of an elementary Euler-Bernoulli problem, provided that edge effects are considered to derive the proper boundary conditions. So we can claim, as others[2,10], that considering end effects makes the use of a refined beam theory unnecessary.

The main interest of the method presented here is to provide the full outer expansion from the numerical solution of only two 2D-problems and a set of 3D canonical problems. Especially, the edge effects are taken into account without computing any inner expansion.

The example of a layered beam shows that the method leads to the approximation of the exact 3D outer solution with a very good accuracy. Applications of this method to the study of thin-walled composite beams will be made in the near future.

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