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New Algorithm for Detecting Weak Changes in the Mean in a Class of CHARN Models with Application to Welding Electrical Signals †

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Abstract: In this paper, we propose a new automatic algorithm for detecting weak changes in the mean of a class of piece-wise CHARN models. Through a simulation experiment, we demonstrate its efficacy and precision in detecting weak changes in the mean and accurately estimating their locations. Furthermore, we illustrate the robust performance of our algorithm through its application to welding electrical signals (WES).

Keywords: change-points; CHARN models; LAN; likelihood-ratio tests; welds

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1. Introduction

The analysis of structural change-points, or breaks, has begun with Page [1] in quality control, but over time, it has expanded to include a strong statistical component in various fields, such as economics (Perron et al. [2]), climatology (Reeves, Chen, Wang, Lund, and Lu [3] and Beaulieu, Chen, and Sarmiento [4]), finance (Andreou and Ghysels [5]), and engineering (Stoumbos, Reynolds Jr, Ryan, and Woodall [6]). The characteristics of changes in time series vary depending on the magnitude of the changes. These changes can manifest prominently (indicative of a substantial magnitude of change) or remain inconspicuous (suggesting a weak magnitude). Furthermore, even when the magnitude is significant, such changes may transpire only over a limited number of observations, referred to in this paper as a "false alarm" or an anomaly in the data. Conversely, when breaks persist over an extended period, the data assume the characteristics of piece-wise stationary data, which is the focus of our investigation in this study.

The literature on change-points is large and varied. Depending on whether the data are given in advance (off-line) or acquired sequentially (on-line). One of the statistics most often used for the segmentation of the time series is the CUSUM test, introduced by Page [1]. Brown, Durbin, and Evans [7] introduce another version of the CUSUM test based on the least-squares residuals, denoted by CUSUM^{ols}. Zeileis [8] and Zeileis [9] use the CUSUM test in order to estimate the p-value. Aue and Horváth [10] show how procedures based on the popular cumulative sum, CUSUM, statistics can be modified to work for data exhibiting serial dependence. In the context of time series, very little is done about testing no change against local alternatives to weak changes. We mean by weak changes those of small magnitudes. Ltaifa [11] and Ngatchou-Wandji and Ltaifa [12] study this problem for the case of testing the mean of Conditional Heteroscedastic Autoregressive Nonlinear "CHARN" model . Salman et al. [13] extends the work of Ltaifa [11] to more general models.

In this paper, we use the theoretical results obtained in Salman et al. [13] and introduce a new algorithm for detecting weak changes in the mean. We examine the performance of the proposed algorithm using simulated data, and we apply it to a real data set such as welding electrical signals (WES). The new algorithm is motivated by the reduction of the effect of the white noise, which can sometimes it be detected by several methods as a change-point. At the same time, the proposed one contains some techniques for identifying the type of changes detected and the distinction between an anomaly (false alarm) and a true change-point.

This paper is categorized as follows. In Section 2, we recall the essential theoretical results presented in Salman et al. [13]. In Section 3, we introduce our new algorithm. In Section 4, a simulation experiment is conducted for the application of our algorithm. In Section 5, an application to a real data set is performed, and Section 6 concludes the paper.

2. Model, Problematic, and Main Results of Salman et al. [13]

In this section, we recall, in a brief way, the method developed in Salman et al. [13], from which it has been constructed for detecting weak changes in the mean based on the theoretical power of the likelihood ratio test. The class of the statistical model presented in Salman et al. [13] is the Conditional Heteroscedastic Autoregressive Nonlinear model "CHARN" (see, e.g., Härdle, Tsybakov, and Yang [14]).

More precisely, let $d, p, k, n \in \mathbb{N}$ and $k << n$. Assume the observations $X_1, \ldots, X_n$ issued from the following piece-wise stationary CHARN model

$$X_t = T(\boldsymbol{\rho}_0 + \boldsymbol{\gamma} \odot \omega(t); \mathbf{X}_{t-1}) + V(\mathbf{X}_{t-1})\varepsilon_t, t \in \mathbb{Z}, \quad (1)$$

with

$$X_t = Y_{t,j} = T(\boldsymbol{\rho}_0 + \gamma_j \omega_j(t); \mathbf{X}_{t-1,j}) + V(\mathbf{X}_{t-1,j})\varepsilon_t, \quad \tau_{j-1} \leq t < \tau_j, \quad j = 1, \ldots, k+1, \quad (2)$$

where for $j = 1, \ldots, k$, $(Y_{t,j})_{t \in \mathbb{Z}}$ is a stationary and ergodic process; $\boldsymbol{\rho}_0 \in \mathbb{R}^p$, $T(\boldsymbol{\rho}_0, .)$ and $V(.)$ are real-valued functions with $\inf_{x \in \mathbb{R}^d} V(x) > 0$; the τ_j, $j = 0, \ldots, k+1$, are potential instants of changes with $\tau_0 = 1$ and $\tau_{k+1} = n+1$; for $j = 1, \ldots, k$, $\mathbf{X}_{t,j} = (Y_{t,j}, \ldots, Y_{t-d+1,j})^\top$, $\mathbf{X}_{\tau_{j-1}+\ell} = \mathbf{X}_{\tau_{j-1}+\ell,j}$, $\ell = 0, \ldots, d-1$ and for $t \in [\tau_{j-1} + d - 1, \tau_j)$, $\mathbf{X}_t = (X_t, \ldots, X_{t-d+1})^\top$; for $j, \ell = 1, \ldots, k$, $j \neq \ell$, the process $(Y_{t,j})_{t \in \mathbb{Z}}$ and $(Y_{t,\ell})_{t \in \mathbb{Z}}$ are mutually independent (Yau and Zhao [15] noted that this assumption can be extended to some weak dependence assumption); $(\varepsilon_t)_{t \in \mathbb{Z}}$ is a standard white noise with density f. $\boldsymbol{\gamma} = \left(\gamma_1^\top, \ldots, \gamma_{k+1}^\top\right)^\top$, $\gamma_j \in \mathbb{R}^p$, $j = 1, \ldots, k+1$; $\omega(t) = (\mathbb{1}_{[\tau_0,\tau_1)}(t), \mathbb{1}_{[\tau_1,\tau_2)}(t), \ldots,$ $\mathbb{1}_{[\tau_{k-1},\tau_k)}(t), \mathbb{1}_{[\tau_k,\tau_{k+1})}(t))^\top = (\omega_1(t), \ldots, \omega_{k+1}(t)) \in \{0,1\}^{k+1}$; for $\boldsymbol{\gamma} = (\gamma_1^\top, \ldots, \gamma_{k+1}^\top)^\top$ and $\omega(t) = (\omega_1(t), \ldots, \omega_{k+1}(t))^\top$, $\boldsymbol{\gamma} \odot \omega(t)$ stands for $\boldsymbol{\gamma} \odot \omega(t) = \gamma_1 \omega_1(t) + \cdots + \gamma_{k+1}\omega_{k+1}(t) \in \mathbb{R}^p$, and $\gamma_i \omega_i = (\gamma_{i,1}\omega_i, \ldots, \gamma_{i,p}\omega_i) \in \mathbb{R}^p$.

This category of models is expansive, encompassing a variety of models including AR(p), ARCH(p), EXPAR(p), GEXPAR(p). Statistical and probabilistic properties have been extensively investigated in the existing literature (see, e.g., Chen, Gan, and Chen [16] for the study of the ergodicity of GEXPAR models).

For $\gamma_0 \in \mathbb{R}^{p(k+1)}$ and $\boldsymbol{\beta} \in \mathbb{R}^{p(k+1)}$ depending on the τ_j's, Salman et al. [13] construct a likelihood ratio test for testing

$$H_0 : \boldsymbol{\gamma} = \boldsymbol{\gamma}_0 \quad \text{against} \quad H_\beta^{(n)} : \boldsymbol{\gamma} = \boldsymbol{\gamma}_n = \boldsymbol{\gamma}_0 + \frac{\boldsymbol{\beta}}{\sqrt{n}}. \quad (3)$$

Note that the norm of $\boldsymbol{\beta}$ is small in front of n, and then the two hypotheses considered are getting closer as the sample size n grows up.

First, the authors prove that the test constructed establish the locally asymptotically normal property (LAN) and the hypotheses considered are contiguous in the sens of Le Cam (see Le Cam [17] and Droesbeke and Fine [18]). These properties allow the study of the theoretical power of the test constructed and lead to obtain an explicit expression of it. Indeed, under some technical hypotheses, they prove that the constructed likelihood ratio test is asymptotically optimal and its asymptotic power has the following expression

$$\mathcal{P}_{k,\tau^k} = 1 - \Phi(z_\alpha - \vartheta(\rho_0, \gamma_0, \beta)) \tag{4}$$

where

- $\boldsymbol{\rho}_0$ represent the true nuisance parameter and $\alpha \in (0,1)$ represent the level of significance,
- z_α is the $(1-\alpha)$-quantile of the standard Gaussian distribution with cumulative distribution function ϕ,
- ϑ is a real function defined in $\mathbb{R}^{p(k+1)\times p(k+1)}$, where its expression is given in Salman et al. [13].

In practice, the parameters are unknown and have to be estimated. Many works focus on the estimation of the parameters; for example, Chen, Gan, and Chen [16] discuss the estimation of the parameters of the linear and non-linear parts in GExpAR models, which are particular cases of the CHARN model studied in [13], Brockwell, Davis, and Salehi [19] for linear models as ARMA, and many others. A decision for the testing problem considered in Salman et al. [13] can be taken to be the estimation of the test's power $\widehat{\mathcal{P}}_{k,\tau^k}$, which is the one obtained by replacing the true parameters with their estimators in $\mathcal{P}_{k,\tau^k}$. To explain the techniques used here for parameter estimation, for $1 \leq j \leq k+1$, $1 \leq h \leq p$, let $\widehat{\rho}_{j,h}$ a consistent estimator (for example, the maximum likelihood estimator) of $\rho_{0,h} + \beta_{j,h}/\sqrt{n}$ on the basis of observations within $[\tau_{j-1}, \tau_j)$. Then one can consider $\widehat{\beta}_{j,h} = \sqrt{n}(\widehat{\rho}_{j,h} - \widehat{\rho}_{0,h})$ as an estimator of $\beta_{j,h}$, where $\widehat{\rho}_{0,h}$ is the estimator of the stationary parameter $\rho_{0,h}$ on the basis of the first piece of observation $[1, \tau_1)$. By replacing the parameters with their estimators, the authors prove that the test constructed remains asymptotically optimal, and they derive an explicit expression of its power, as noted by $\widehat{\mathcal{P}}_{k,\tau^k}$.

3. New Algorithm for Weak-Changes Detection and Their Locations Estimation

Here, we introduce a new algorithm motivated by both the reduction of the impact of white noise and the classification of the detected changes into change-points and false alarms. In the sequel, we denote by $\mathcal{P}_{k,\tau^k}$, $k \geq 1$ the theoretical power of the test considered at $\tau^k = (\tau_1, \ldots, \tau_k)$. For $\alpha \in (0,1)$ representing the level of significance, we denote by $\mathcal{P}_{0,\tau^0} = \alpha$ the nominal level of the test.

Let $\zeta \in (0, .1)$ and $X_1, X_2, \ldots, X_m$, $(m << n)$, the m first stationary observations. A crucial point to mention is that, in practice, m will be smaller than that considered in [13]. Our procedure for detecting weak changes in the time series $X_1, X_2, \ldots, X_n$ and estimating their locations is described in the following Algorithm 1.

Algorithm 1 Automatic algorithm for weak changes detection

Location 1 :

Put $t = 1$

$(\mathcal{S}_1)$: Consider the two intervals $\mathcal{I}_1$ and $\mathcal{I}_2$ that contains respectively the observations $X_1, \ldots, X_{m+t-1}$ and $X_1, \ldots, X_{m+t}$. So that the difference between the two intervals considered is the single observation X_{m+t} which is under testing.

$(\mathcal{S}_1)'$: **Adjust** model (1) to $\mathcal{I}_1$ and $\mathcal{I}_2$. Then, apply the testing procedure presented in [13].

If $|\mathcal{P}_{1,t} - \mathcal{P}_{0,\tau^0}| > \zeta$,

Replace X_{m+t} with $X_{m+\varsigma}$ in $\mathcal{I}_2$, with $t+1 \leq \varsigma \leq j$, $j << m$, and **Repeat** $(\mathcal{S}_1)'$ with the updated $\mathcal{I}_2$

If $|\mathcal{P}_{1,\varsigma} - \mathcal{P}_{0,\tau^0}| > \zeta$,

The first change location is estimated on $\tau_1 = m + t$.

Then, Go to **Location 2**.

Else

A **False Alarm** is detected.

Remove X_{m+t} from the sample, Do $t = t + 1$ and Go to $(\mathcal{S}_1)$.

Else

Do $t = t + 1$ and Go to $(\mathcal{S}_1)$.

Location i :

We already estimated the $(i-1)^{th}$ change location τ_{i-1} in step $i - 1$

Consider the next h observations to $X_{\tau_{i-1}}$: $X_{\tau_{i-1}+1}, \ldots, X_{\tau_{i-1}+h}$

Put $t = 1$ and Do

$(\mathcal{S}_i)$: Consider the two intervals $\mathcal{I}_1$ and $\mathcal{I}_2$ that contains respectively the observations $X_{\tau_{i-1}}, \ldots, X_{\tau_{i-1}+h+t-1}$ and $X_{\tau_{i-1}}, \ldots, X_{\tau_{i-1}+h+t}$. So that the difference between the two intervals considered is the single observation under testing.

$(\mathcal{S}_i)'$: **Adjust** model (1) to $\mathcal{I}_1$ and $\mathcal{I}_2$. Then, apply the testing procedure presented in ().

If $|\mathcal{P}_{1,t} - \mathcal{P}_{0,\tau^0}| > \zeta$,

Replace $X_{\tau_{i-1}+h+t}$ with $X_{\tau_{i-1}+h+\varsigma}$ in $\mathcal{I}_2$, with $t+1 \leq \varsigma \leq j$, $j << h$, and **Repeat** $(\mathcal{S}_i)'$ with the updated $\mathcal{I}_2$

If $|\mathcal{P}_{1,\varsigma} - \mathcal{P}_{0,\tau^0}| > \zeta$,

The i^{th} change location is estimated on $\tau_i = \tau_{i-1} + h + t$.

Then, Go to **Location** $i + 1$.

Else

A **False Alarm** is detected.

Remove $X_{\tau_{i-1}+h+t}$ from the sample, Do $t = t + 1$ and Go to $(\mathcal{S}_i)$.

Else

Do $t = t + 1$ and Go to $(\mathcal{S}_i)$.

4. Simulation Experiment

For the simulation, we use the same particular CHARN model as in Salman et al. [13] having the following expression

$$X_t = \rho_{0,1} + \frac{\beta_{j,1}}{\sqrt{n}} + \left(\rho_{0,2} + \frac{\beta_{j,2}}{\sqrt{n}}\right) X_{t-1}\, e^{\left(\rho_{0,3} + \frac{\beta_{j,3}}{\sqrt{n}}\right) X_{t-1}^2} + \sqrt{\theta_1 + \theta_2 X_{t-1}^2}\, \varepsilon_t, \tag{5}$$

$j = 1, \ldots, k, \quad t \in \mathbb{Z},$

where n denotes the number of observations, $(\varepsilon_t)_t$ is a standard white noise with a differentiable density f. Here, on $[\tau_{j-1}, \tau_j)$, $\rho_0 = (\rho_{0,1}, \rho_{0,2}, \rho_{0,3}) \in \mathbb{R}^3$, $\beta_j = (\beta_{j,1}, \beta_{j,2}, \beta_{j,3}) \in \mathbb{R}^3$; ρ_0 is the parameter to be specified in each particular model considered.

Data Presenting One Single False Alarm

In this section, we consider the problem of detecting and identifying the change faced. The data are generated by model (5) for $n = 300$, $\rho_{0,1} = 0.2$, $\rho_{0,2} = 0.3$, $\rho_{0,3} = 0$, $\beta_{1,1} = 5$, $\beta_{1,2} = -3$, $\beta_{j,3} = \theta_2 = 0$ and $\theta_1 = 1$. At an instant between 1 and n, say τ_2, we replace the corresponding observation, say X_{τ_2}, by another observation, for example, ϵ where $\epsilon \sim \mathcal{N}(-1, 2)$. For $\alpha = 5\%$ and $\zeta = 0.25\%$, Figure 1 illustrates the behavior of the power when facing a change and Table 1 shows the results obtained for different values of ζ.

Table 1. Power around changes detected in a class of AR(1) model.

	$((\beta_{11}, \beta_{1,2}), (\tau_1, \tau_2), \epsilon \sim, \zeta))^\top$			
$\widehat{\tau}$ and Power	$(1,1)$ $(101,200)$ $\mathcal{N}(1,1)$ 0.15%	$(3,-2)$ $(101,250)$ $\mathcal{N}(1,2)$ 0.25%	$(5,-3)$ $(111,280)$ $\mathcal{N}(-1,2)$ 0.25%	$(10,-6)$ $(91,295)$ $\mathcal{N}(2,2)$ 0.35%
$\widehat{\tau}_1$	102	101	111	91
$\widehat{\mathcal{P}}_{1,\tau_1-1}$	0.050541	0.050713	0.050811	0.050972
$\widehat{\mathcal{P}}_{1,\tau_1}$	0.052418	0.053712	0.054612	0.057321
$\widehat{\mathcal{P}}_{1,\tau_1+1}$	0.052503	0.053515	0.054874	0.057819
$\widehat{\mathcal{P}}_{1,\tau_1+2}$	0.052315	0.053821	0.054731	0.057643
$\widehat{\mathcal{P}}_{1,\tau_1+3}$	0.052517	0.053644	0.054912	0.057967
$\widehat{\mathcal{P}}_{1,\tau_1+4}$	0.052421	0.053553	0.054826	0.058042
$\widehat{\tau}_2$	200	250	280	295
$\widehat{\mathcal{P}}_{2,\tau_2-1}$	0.050912	0.050626	0.050963	0.050121
$\widehat{\mathcal{P}}_{2,\tau_2}$	0.052915	0.054261	0.053987	0.061092
$\widehat{\mathcal{P}}_{2,\tau_2+1}$	0.051981	0.051725	0.516471	0.051681
$\widehat{\mathcal{P}}_{2,\tau_2+2}$	0.051734	0.051628	0.051811	0.051874
$\widehat{\mathcal{P}}_{2,\tau_2+3}$	0.051413	0.051632	0.051736	0.051642
$\widehat{\mathcal{P}}_{2,\tau_2+4}$	0.051386	0.051589	0.051481	0.051328

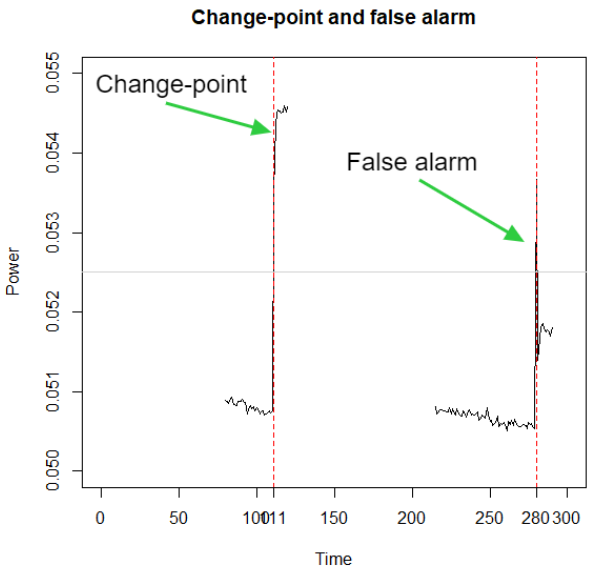


Figure 1. Behavior of the power when facing a change.

5. Welding Electrical Signals

Here, using our algorithm mentioned in Section 3, we apply the method of [13] for detecting weak changes in the mean of arc-welding series. First, the chronogram of the WES series (W_t) seems to present a trend and does not present a seasonality. The Augmented Dicky-Fuller test (see Cheung and Lai [20]) approve the non-stationarity of this series. For that, we decompose this series in a summation of two components as follow:

$$W_t = Y_t + X_t,$$

where (Y_t) represents the unknown trend assumed to be continuous and (X_t) is a piecewise stationary series with mean (μ_t) and variance (σ_t). Using the Akaike Information criterion AIC (see [21]), we estimate the trend by the following moving-average with order 5

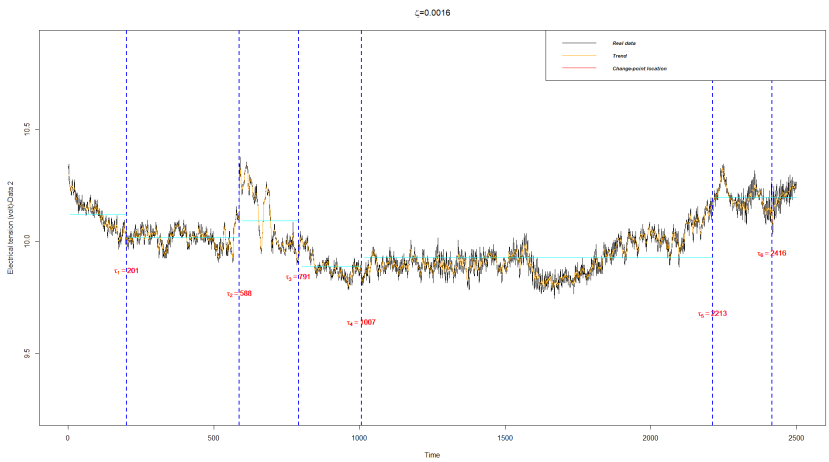
$$\widehat{Y}_t = \frac{1}{5} \sum_{j=-2}^{2} W_{t+j}.$$

The Box-Ljung and Box-Pierce tests (see Brockwell and Davis [22]) applied to the residual series reject the null hypotheses, and then they are not iid. Also, the QQ-plot and the histogram of the residuals seems to explain that the residual series is normally distributed in addition to Shapiro-Wilk test. Basing on all of these investigations, we assume the heteroscedasticity of the residual series and by taking into consideration the AIC, we propose a shifted model defined as follow:

$$X_t = \rho_{0,1} + \frac{\beta_{j,1}}{\sqrt{n}} + \sigma_j \varepsilon_t, \quad t \in [\tau_{j-1}, \tau_j), \quad j = 1, \ldots, k+1,$$

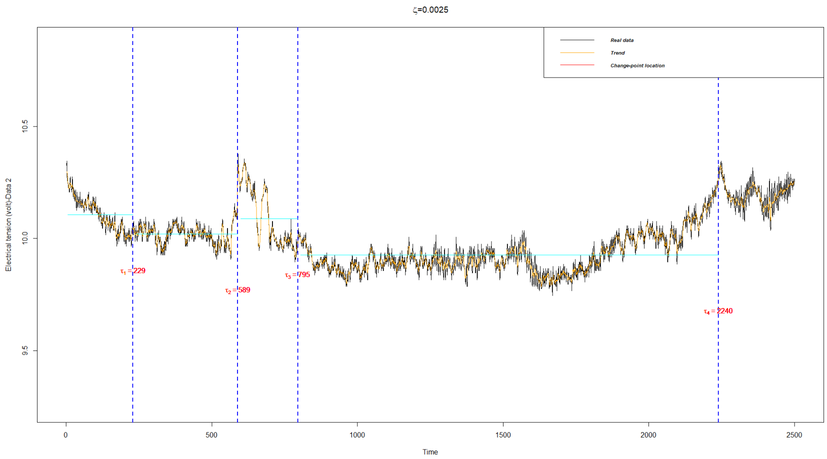
where k is the number of change-points which is unknown and must be estimated, $\tau_1, \ldots, \tau_k$ designate the change-point locations, (ε_t) is a standard Gaussian white noise, $V(x) = \sigma_j$ represents the variance of X_t in each interval $[\tau_{j-1}, \tau_j)$.

Here, for the test problem, $\gamma_0 = 0$, $\gamma_n = (0, \beta_2/\sqrt{n}, \ldots, \beta_{k+1}/\sqrt{n}) \in \mathbb{R}^{k+1}$ with $\boldsymbol{\beta} = (0, \beta_2, \ldots, \beta_{k+1})$. Figure 2 shows the breaks detected in the WES series for $\zeta = 0.15\%$, and 0.25%. This series is considered as a normal weld, and we can mention that for $\zeta = 3\%$, no change has been detected.



(**a**) $\zeta = 0.15\%$.

Figure 2. *Cont.*



(**b**) $\zeta = 0.25\%$.

Figure 2. Change-points estimates in WES corresponding to two different thresholds.

6. Conclusions

We have introduced a new algorithm for detecting weak changes in the mean using the method proposed by Salman et al. [13]. The simulation experiment conducted shows that our algorithm is efficient in detecting multiple breaks and distinguishing between a change-point and a false alarm. Comparing to the results obtained in Salman et al. [13], our algorithm seems to be more accurate.

A facet of our perspective, relevant to this research, entails creating an automated method to ascertain the optimal threshold appropriate for the particular domain under investigation. Tackling this worldwide challenge is a crucial consideration for myriad researchers in this discipline, and it serves as a substantial focal point for our upcoming endeavors.

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